

PERTINENCY OF HOPF ACTIONS AND QUOTIENT CATEGORIES OF COHEN-MACAULAY ALGEBRAS

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ABSTRACT. We study invariants and quotient categories of fixed subrings of Artin-Schelter regular algebras under Hopf algebra actions.

0. INTRODUCTION

Artin-Schelter regular (or AS regular, for short) algebras [Definition 1.8] play an important role in noncommutative algebraic geometry. We study some classes of graded algebras that are closely related to AS regular algebras. In particular, we are interested in algebraic and homological properties of fixed subrings of AS regular algebras under semisimple Hopf algebra actions, or geometrically, properties of quotient singularities of noncommutative spaces under quantum group actions. Some homological properties of Hopf actions on AS regular algebras were studied in [KKZ1]. Finite dimensional Hopf actions on connected graded AS regular algebras of global dimension two with trivial homological determinant were classified in [CKWZ1]. In this paper we further investigate Hopf actions on AS regular algebras and the correspondence quotient singularities in the noncommutative setting.

One of most interesting features in this area is the *McKay correspondence*. The classical McKay correspondence connects many different topics in mathematics such as the finite subgroups of $SU(2, \mathbb{C})$, Dynkin diagrams of *ADE* type, simple Lie algebras, Kleinian singularities, *G*-Hilbert schemes and so on. See [Bu, Le1, Le2] for very nice recent surveys and historical account. One ingredient in the classical or commutative McKay correspondence is the Auslander Theorem [Au1, Au2] that states that, *if G is a finite small subgroup of $GL(\mathbb{C}^{\oplus n})$ acting on $R := \mathbb{C}[x_1, \dots, x_n]$ naturally, then there is a natural isomorphism of algebras*

$$(E0.0.1) \quad R * G \cong \text{End}_{R^G}(R).$$

Recall that a finite subgroup $G \subseteq GL(V)$, for a finite dimensional vector space V , is called *small* if G does not contain a pseudo-reflection of V .

Different extensions and noncommutative versions of the McKay correspondence have been proposed by several researchers [Ch, CKWZ2, Mor2, Le2, Wem] and many nice results have been proven. In noncommutative algebraic geometry, AS regular algebras are correct noncommutative analogues of the commutative polynomial rings. One key ingredient in noncommutative McKay correspondence is the Auslander Theorem when R is replaced by a noncommutative AS regular algebra.

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It is desirable to prove the Auslander Theorem in a general noncommutative setting. Some partial results can be found in [BHZ, CKWZ2, HVZ2, MU] and so on, under some extra hypotheses. One of the main goals of this paper is to prove a noncommutative version of the Auslander theorem that recovers the most of the previous results and to provide some conjectures in this research direction.

Let A be an \mathbb{N} -graded algebra over a base field \mathbb{k} of characteristic zero. We are taking one step further by replacing the group G in (E0.0.1) by a Hopf algebra H . Let H be a nontrivial semisimple Hopf algebra acting on A homogeneously and inner faithfully [Definition 3.9]. Let $A \# H$ denote the smash product [Mon, Definition 4.1.3]. Since H is semisimple, the left and the right integrals of H coincide, and is denoted by \int . The main *dimension function* we are going to use here is the Gelfand-Kirillov dimension (also denoted by GK-dimension) – see [KL] or Definition 1.1. A new invariant we introduce in this paper is the following.

Definition 0.1. Retain the above notation and suppose that $\text{GKdim } A < \infty$. The *pertinency* of the H -action on A is defined to be

$$\mathfrak{p}(A, H) = \text{GKdim } A - \text{GKdim}((A \# H)/I)$$

where I is the 2-sided ideal of $A \# H$ generated by $1 \# \int$.

Fix an A and an H , different H -actions on A result different pertinency. This means that $\mathfrak{p}(A, H)$ is an invariant of the H -action on A , not just the pair (A, H) . When H is the group algebra $\mathbb{k}G$, we write $\mathfrak{p}(A, H)$ as $\mathfrak{p}(A, G)$. If the H -action on A was not inner faithful, then it is easy to check that $\mathfrak{p}(A, H) = 0$. If H acts on a domain A inner faithfully and $A \# H$ is prime, it is automatic that $\mathfrak{p}(R, H) \geq 1$ (see Lemma 3.10). Note that $\mathfrak{p}(A, H) = \text{GKdim } A$ (or equivalently, $\text{GKdim}((A \# H)/I) = 0$) if and only if A/A^H is a right H^* -dense Galois extension in the sense of [HVZ2].

In part of the paper, we use Cohen-Macaulay property [Definition 1.4(3)] which is a homological property in the noncommutative setting that is intimately connected with the AS regularity. For example, all well-studied noetherian AS regular algebras have been proven to be Cohen-Macaulay and it is conjectured that the Cohen-Macaulay property is a consequence of the AS regularity.

Setup: For the rest of introduction, let R denote a noetherian, connected graded AS regular, Cohen-Macaulay algebra of GK-dimension ≥ 2 .

Definition 0.2. Suppose H is a Hopf algebra with integral \int acting on R inner faithfully, and let M be a finitely generated right R -module.

- (1) The *grade* of M is defined to be

$$j(M) := \min\{i \mid \text{Ext}_R^i(M, R) \neq 0\}.$$

If $\text{Ext}_R^i(M, R) = 0$ for all i , then we say $j(M) = \infty$.

- (2) We say that the H -action on R is *homologically small* if $j(R \# H/(I)) \geq 2$, where I is the 2-sided ideal generated by $1 \# \int$.

In the commutative (graded) case, homological smallness is equivalent to the (usual) smallness [BHZ, Lemma 7.2]. Our first result connects the noncommutative Auslander theorem with the pertinency.

Theorem 0.3. Let H be a semisimple Hopf algebra acting on R inner faithfully and homogeneously. Then the following are equivalent.

- (1) $\mathfrak{p}(R, H) \geq 2$.
- (2) *The H -action on R is homologically small.*
- (3) *There is a natural isomorphism of algebras $R\#H \cong \text{End}_{R^H}(R)$.*

Theorem 0.3 generalizes most of previous results concerning the noncommutative Auslander theorem in [CKWZ2, HVZ2, MU]. It is easier to manipulate the condition on $\mathfrak{p}(R, H)$ than to prove the isomorphism in Theorem 0.3(3) directly, see [BHZ]. This is an advantage of Theorem 0.3. In Theorems 0.3 and 0.6, the Cohen-Macaulay property is crucial while the AS regularity is hidden in the background. Then, in Theorem 0.7, the AS regularity is definitely needed. By the above theorem, checking the hypothesis on the pertinency “ $\mathfrak{p}(R, H) \geq 2$ ” becomes essential. In some cases, the inequality $\mathfrak{p}(R, H) \geq 2$ is checkable – see Theorem 0.5 and [BHZ]. We do not have a general algorithm of calculating this number. However, the following conjecture is expected.

Conjecture 0.4. *Retain the hypotheses as in Theorem 0.3. Suppose the H -action on R has trivial homological determinant in the sense of [KKZ1, Definition 3.3]. Then $\mathfrak{p}(R, H) \geq 2$, or equivalently, the H -action on R is homologically small.*

When R is the commutative polynomial ring and H is a finite group algebra, this conjecture holds as a consequence of the classical Auslander theorem. In the noncommutative setting, this has been verified in some special cases, but the conjecture is open in general. If Conjecture 0.4 holds, then, by combining with Theorem 0.3, we have the following version of the Auslander theorem: *Retain the hypotheses as in Theorem 0.3 and further assume that the H -action on R has trivial homological determinant. Then $R\#H \cong \text{End}_{R^H}(R)$ naturally.*

Note that the aim of paper [CKWZ2] is to establish a noncommutative McKay correspondence for Hopf algebra actions on AS regular algebras of dimension two, and one of the main results there is to verify Conjecture 0.4 when $\text{gldim } R = 2$.

As a consequence of Theorem 0.3, we have the following result. For any graded algebra A , let $\text{Aut}_{gr}(A)$ denote the group of all graded algebra automorphisms of A . Suppose $n \geq 2$ and let $R := \mathbb{k}_{-1}[x_1, \dots, x_n]$ be the (-1) -skew polynomial ring generated by x_1, \dots, x_n and subject to the relations $x_i x_j = -x_j x_i$ for all $i \neq j$. Then $\text{Aut}_{gr}(R) = (\mathbb{k}^\times)^n \rtimes S_n$ [KKZ2, Lemma 1.12(1)].

Theorem 0.5. *Let σ be the automorphism of $R := \mathbb{k}_{-1}[x_1, \dots, x_n]$ determined by sending $x_i \rightarrow x_{i+1}$ for all $1 \leq i \leq n-1$ and $x_n \rightarrow x_1$. Let G be the group generated by σ . Then the homological determinant of the G -action on R is trivial and $\mathfrak{p}(R, G) \geq 2$. As a consequence, $R * G \cong \text{End}_{R^G}(R)$.*

Theorem 0.5 answers a question of Ellen Kirkman [Ki], see also Question 5.9. In contrast, when R is a commutative polynomial ring $\mathbb{k}[x_1, \dots, x_n]$ and G is generated by the permutation sending $x_i \rightarrow x_{i+1}$ for all $1 \leq i \leq n-1$ and $x_n \rightarrow x_1$, then

$$\mathfrak{p}(R, G) = \begin{cases} 1 & n = 2, \\ \geq 2 & n > 2. \end{cases}$$

So, in the commutative case, G is not (homologically) small when $n = 2$, see [BHZ, Examples 7.3 and 7.4] for further comments.

Going back to a general (graded) algebra A , we write $\text{Gr } A$ for the category of right graded A -modules, and $\text{gr } A$ for the subcategory of finitely generated right

graded A -modules. For any integer n , let $\text{Gr}_n A$ (respectively, $\text{gr}_n A$) be the subcategory of $\text{Gr } A$ (respectively, $\text{gr } A$) consisting of objects with GK-dimension not larger than n . Define the following quotient categories

$$\text{QGr}_n B := \frac{\text{Gr } B}{\text{Gr}_n B} \quad \text{and} \quad \text{qgr}_n B := \frac{\text{gr } B}{\text{gr}_n B}.$$

When $n = 0$, the quotient category $\text{QGr}_0 B$ (respectively, $\text{qgr}_0 B$) is just $\text{QGr } B$ (respectively, $\text{qgr } B$) defined in noncommutative projective geometry [AZ] or $\text{Tails } B$ (respectively, $\text{tails } B$) used by other authors [MU, Ue, Mor1, Mor2].

Our first result relating the pertinency and the quotient category is the following.

Theorem 0.6. *Retain the hypotheses as in Theorem 0.3. Let α be an integer $\leq \mathfrak{p}(R, H)$. Then there is an equivalence of categories*

$$\text{qgr}_{n-\alpha} R^H \cong \text{qgr}_{n-\alpha} R \# H.$$

Since R is AS regular, $R \# H$ has finite global dimension. If, in addition, R is PI (namely, R satisfies a polynomial identity), then the quotient category $\text{qgr}_{n-\alpha} R \# H$ has finite global dimension. Under the extra PI hypothesis, by the above theorem, $\text{qgr}_{n-\alpha} R^H$ has finite global dimension. When a connected graded algebra A is PI, it is reasonable to define the dimension of the singular locus of A to be

$$\dim_{\text{sing}}(A) = \min\{i \mid \text{qgr}_i A \text{ has finite global dimension}\}.$$

Using this definition and assuming R is PI, we can bound the \dim_{sing} of the fixed subring R^H by

$$(E0.5.1) \quad \dim_{\text{sing}}(R^H) \leq n - \mathfrak{p}(R, H) = \text{gldim } R - \mathfrak{p}(R, H).$$

In particular, when $\mathfrak{p}(R, H) = \text{gldim } R$, R^H has graded isolated singularities in the sense of [Ue, Definition 2.2]. We give new examples of noncommutative graded isolated singularities in Proposition 5.4. Following the idea of the noncommutative crepant resolutions [Le2], $R \# H$ should be viewed a resolution of singularities of R^H . Some ideas in this paper are motivated by recent papers [CKWZ1, CKWZ2, Le2, Mor1, Mor2, MU, HVZ2, Ue].

Another result concerning quotient categories is a version of BGG correspondence. Let E be the graded algebra

$$E := \bigoplus_{i=0}^n \text{Ext}_{R \# H}^i(S, S)$$

where S is a direct sum of all non-isomorphic graded simple right modules over $R \# H$ concentrated in degree zero. Since R is Koszul, so is $R \# H$. As a consequence, E is Koszul. (All these are \mathbb{N} -graded and locally finite.) Since R is AS regular, E is Frobenius. Then the stable module category $\underline{\text{gr}} E$ is a triangulated category. Let M be a finite dimensional right graded E -module. Take a minimal graded projective resolution of M as follows

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0.$$

The *complexity* of M , denoted by $c(M)$, is the least integer $\lambda \geq 0$ such that $\dim P^{-n} < a n^{\lambda-1}$ for almost all n , where $a > 0$ is a fixed number. The complexity of a module over a finite dimension algebra is an important invariant in studying the representations of finite dimensional algebras [Ca, CDW, GLW]. Given any nonnegative integer β , let \mathcal{C}_β be the full subcategory of $\text{gr } E$ consisting of objects M with $c(M) \leq \beta$.

Theorem 0.7. *Retain the hypotheses as in Theorem 0.3 and assume further that R is Koszul and that \mathbb{k} is algebraically closed. Let β be a nonnegative integer.*

- (1) *There is a canonical anti-equivalences of triangulated categories*

$$D^b(\text{qgr}_\beta R \# H) \cong \underline{\text{gr}} E / \mathcal{C}_\beta.$$

- (2) *For any $\beta \geq n - \mathbf{p}(R, H)$, there is an canonical anti-equivalence of triangulated categories*

$$D^b(\text{qgr}_\beta R^H) \cong \underline{\text{gr}} E / \mathcal{C}_\beta.$$

When H is a group algebra, one can define another invariant associated to the G -action on R . Let now R be as in Theorem 0.7. Then the global dimension and GK-dimension of R are the same. For any graded algebra automorphism g of R , one can define the trace function $Tr(g, t)$ as in [KKZ2, Section 1].

Definition 0.8. Let g be a graded algebra automorphism of R of finite order and let G be a finite subgroup of $\text{Aut}_{gr}(R)$.

- (1) The *reflection number* of g is defined to be

$$\mathbf{r}(g) = \text{GKdim } R - \text{the order of the pole of } Tr(g, t) \text{ at } t = 1.$$

- (2) [KKZ3, Definition 2.2] g is called a *quasi-reflection* if $\mathbf{r}(g) = 1$.
(3) [KKZ4, Definition 3.6(b)] g is called a *quasi-bireflection* if $\mathbf{r}(g) = 2$.
(4) The *reflection number* of G -action on R is defined to be

$$\mathbf{r}(R, G) = \min\{\mathbf{r}(g) \mid 1 \neq g \in G\}.$$

- (5) G is called *conventionally small* if $\mathbf{r}(R, G) \geq 2$, or equivalently, G does not contain any quasi-reflection.

Similar to $\mathbf{p}(R, G)$, $\mathbf{r}(R, G)$ is an invariant of the G -action on R , not of the pair (R, G) . The relation between $\mathbf{p}(R, G)$ and $\mathbf{r}(R, G)$ is not straightforward, and what we are hoping for is the following conjectural inequality.

Conjecture 0.9. *Retain the hypotheses as in Theorem 0.7. Let G be a finite subgroup of $\text{Aut}_{gr}(R)$. Then*

$$(E0.8.1) \quad \mathbf{p}(R, G) \geq \mathbf{r}(R, G).$$

This conjecture can be verified for some special cases [Theorem 5.7(4)]. If it holds, then Conjecture 0.4 follows when H is a group algebra because $\mathbf{r}(R, G) \geq 2$ when G -action has trivial homological determinant. In fact, if Conjecture 0.9 holds, then, by combining with Theorem 0.3, we obtain the following: *Retain the hypotheses as in Theorem 0.7 and further assume that the G is conventionally small. Then $R * G \cong \text{End}_{R^G}(R)$ naturally.*

Combining (E0.5.1) with (E0.8.1) we have another conjecture

$$(E0.8.2) \quad \dim_{\text{sing}}(R^G) \leq \text{gldim } R - \mathbf{r}(R, G).$$

The advantage of this conjectural inequality is that, in contrast to $\mathbf{p}(R, G)$, the invariant $\mathbf{r}(R, G)$ is relatively easy to calculate. Note that (E0.8.1) can be verified when R is AS regular of global dimension two following the classification of all such G -actions in [CKWZ1]. When the G -action on R has trivial homological determinant, then $\mathbf{p}(R, G)$ (respectively, $\mathbf{r}(R, G)$) would be either 2 or 3 and $\dim_{\text{sing}}(R^G)$ would be either 0 or 1. Understanding all possibilities of $\{\mathbf{p}(R, G), \mathbf{r}(R, G), \dim_{\text{sing}}(R^G)\}$ in dimension three should be an interesting project.

This paper is organized as follows. We provide background material in Section 1. In Section 2, we prove a Morita type theorem for the equivalences of the quotient categories. In Section 3, we consider semisimple Hopf actions on graded Cohen-Macaulay algebras. We prove a generalized version of Theorem 0.3, and then give a proof of Theorem 0.6. In Section 4, we establish a version of BGG correspondence between the derived categories of quotient categories and the stable categories of some Frobenius categories. In particular, we obtain Theorem 0.7. In Section 5, we compute the pertinency of finite group actions on some special algebras, and give a proof of Theorems 0.5.

1. PRELIMINARIES

Throughout let \mathbb{k} be a base ring that is a noetherian commutative domain. All algebraic objects are over \mathbb{k} . An algebra means an associative \mathbb{k} -algebra that is projective as a \mathbb{k} -module. In Sections 4 and 5 (and part of Section 3) we further assume that \mathbb{k} is a field. Let B be a (left and right) noetherian algebra. Usually we are working with right B -modules. We write $\text{Mod } B$ for the category of all right B -modules, and $\text{mod } B$ for the full subcategory of all finitely generated (or f.g. for short) right B -modules.

Several results in this paper concern quotient categories of the module categories defined via a dimension function. For the purpose of this paper, a function

$$\partial : \text{Mod } B \longrightarrow \mathbb{R} \cup \{\pm\infty\}$$

is called a *dimension function* if, for all f.g. B -modules M ,

$$(E1.0.1) \quad \partial(M) \geq \max\{\partial(N), \partial(M/N)\},$$

whenever N is a submodule of M . The ∂ is called an *exact dimension function* if for all f.g. B -modules M ,

$$(E1.0.2) \quad \partial(M) = \max\{\partial(N), \partial(M/N)\}$$

whenever N is a submodule of M . If M is not f.g., we extend the definition of ∂ by

$$(E1.0.3) \quad \partial(M) = \sup\{\partial(N) \mid \text{for all f.g. } B\text{-submodules } N \subseteq M\}.$$

By using (E1.0.3), equation (E1.0.2) holds for all B -modules M as well. Our definition of a dimension function is weaker than the definition given in [MR, 6.8.4], however, (E1.0.2) is the only property we will use.

For a \mathbb{k} -module V , $\dim V$ denotes $\dim_Q(V \otimes_{\mathbb{k}} Q)$ where Q is the fraction field of \mathbb{k} . One special dimension function is the following

Definition 1.1. [KL, Definition 2.1] Let B be an algebra and M a right B -module.

- (1) The *Gelfand-Kirillov dimension* or *GK-dimension* of B is defined to be

$$\text{GKdim } B = \sup_V \left\{ \overline{\lim}_{n \rightarrow \infty} \log_n(\dim V^n) \mid \text{f.g. } \mathbb{k}\text{-submodules } V \subseteq B \right\}.$$

- (2) The *Gelfand-Kirillov dimension* or *GK-dimension* of M is defined to be

$$\text{GKdim } M = \sup_{V, W} \left\{ \overline{\lim}_{n \rightarrow \infty} \log_n(\dim WV^n) \mid \text{f.g. } \mathbb{k}\text{-submodules } V \subseteq B, W \subseteq M \right\}.$$

See [KL] and Remark 1.5 for basic properties of the GK-dimension.

In a major part of the paper the dimension function is chosen to be the Gelfand-Kirillov dimension. However, a general dimension function will be useful when we

deal with the local setting in the sequel [BHZ] in which case the GK-dimension is usually infinite and does not work well.

From now on we fix an exact dimension function ∂ . Given an integer $n \geq 0$, let $\text{Mod}_n B$ denote the full subcategory of $\text{Mod } B$ consisting of right B -modules with ∂ -dimension less than or equal to n , and let

$$\text{mod}_n B = \text{Mod}_n B \bigcap \text{mod } B.$$

Since ∂ is *exact* on right B -modules, $\text{Mod}_n B$ (respectively, $\text{mod}_n B$) is a Serre subcategory of $\text{Mod } B$ (respectively, $\text{mod } B$). Hence it makes sense to define the quotient categories:

$$\text{QMod}_n B := \frac{\text{Mod } B}{\text{Mod}_n B}, \quad \text{and} \quad \text{qmod}_n B := \frac{\text{mod } B}{\text{mod}_n B}.$$

We denote the natural projection functor by

$$\pi : \text{Mod } B \longrightarrow \text{QMod}_n B.$$

For $M \in \text{Mod } B$, we will write \mathcal{M} for the object $\pi(M)$ in $\text{QMod}_n B$. The hom-set of the objects in the quotient category is defined by

$$\text{Hom}_{\text{QMod}_n B}(\mathcal{M}, \mathcal{N}) = \varinjlim \text{Hom}_B(M', N')$$

for $M, N \in \text{Mod } B$, where M' is a submodule of M such that $\partial(M/M') \leq n$, $N' = N/T$ for some submodule T with $\partial(T) \leq n$, and the direct limit runs over all the pairs (M', N') with these properties.

Definition 1.2. Let A and B be noetherian algebras and ∂ be an exact dimension function that is defined on right A -modules and on right B -modules. Let n and i be nonnegative integers. Let ${}_A M_B$ denote one (or any) bimodule which is f.g. both as a left A -module and as a right B -module.

- (1) We say ∂ satisfies $\gamma_{n,i}(M)$ if for any $N \in \text{mod}_n A$, $\text{Tor}_j^A(N, M) \in \text{mod}_n B$ for all $0 \leq j \leq i$.
- (2) We say ∂ satisfies $\gamma_{n,i}$ if it satisfies $\gamma_{n,i}(M)$ for all M given as above.

In the next section we will be particularly interested in the $\gamma_{n,1}$ property.

Lemma 1.3. Let A and B be noetherian algebras such that ∂ is an exact dimension function on both right A -modules and right B -modules. Assume that a bimodule ${}_A M_B$ is f.g. as a left A and as a right B -module and that ∂ satisfies $\gamma_{n,1}(M)$.

- (1) The functor $- \otimes_A M$ induces a functor

$$- \otimes_A \mathcal{M} : \text{qmod}_n A \longrightarrow \text{qmod}_n B.$$

- (2) The functor $- \otimes_A M$ induces a functor

$$- \otimes_A \mathcal{M} : \text{QMod}_n A \longrightarrow \text{QMod}_n B.$$

Proof. We first prove (2).

For $K \in \text{Mod}_n A$, it is a direct limit of f.g. submodules. Since the direct limit is an exact functor and, by assumption, ∂ is exact and satisfies $\gamma_{n,1}(M)$, we obtain that $\partial(K \otimes_A M) \leq n$ and that $\partial(\text{Tor}_1^A(K, M)) \leq n$ for all $K \in \text{Mod}_n A$. Let $f : U \rightarrow V$ be a morphism in $\text{Mod } A$ such that $\ker f$ and $\text{coker } f$ are in $\text{Mod}_n B$. By the right exactness of $- \otimes_A M$, the cokernel of the right B -module morphism

$$f \otimes_A M : U \otimes_A M \rightarrow V \otimes_A M$$

is in $\text{Mod}_n B$. Write f as the composition

$$f = h \circ g : U \xrightarrow{g} W \xrightarrow{h} V.$$

Since $\partial(\text{Tor}_1^A(\text{coker}(f), M)) \leq n$, by the long exact sequence associated to $-\otimes_A M$, $\partial(\ker(h \otimes_A M)) \leq n$. Let $X = \ker f = \ker g$. Then $\partial(X) \leq n$. We have an exact sequence

$$X \otimes_A M \longrightarrow U \otimes_A M \xrightarrow{g \otimes_A M} W \otimes_A M \longrightarrow 0$$

which induces the exact sequence

$$X \otimes_A M \longrightarrow \ker(f \otimes_A M) \longrightarrow \ker(h \otimes_A M) \longrightarrow 0.$$

Since ∂ is exact, we see

$$\partial(\ker(f \otimes_A M)) \leq \max\{\partial(X \otimes_A M), \partial(\ker(h \otimes_A M))\} \leq n.$$

Hence $f \otimes_A M$ has its kernel and cokernel in $\text{Mod}_n B$. Therefore, $-\otimes_A M$ induces a functor

$$-\otimes_A \mathcal{M} : \text{QMod}_n A \longrightarrow \text{QMod}_n B.$$

So we have proven part (2).

Since M is f.g. in both sides, this functor restricts to

$$-\otimes_A \mathcal{M} : \text{qmod}_n A \longrightarrow \text{qmod}_n B.$$

Therefore (1) follows. \square

We now recall a definition.

Definition 1.4. Let B be an algebra and M a right B -module.

- (2) [ASZ1, Definition 0.4] Let ∂ be a dimension function. We say B is ∂ -Cohen-Macaulay (or ∂ -CM) if $\partial(B) = d \in \mathbb{N}$, and

$$j(M) + \partial(M) = \partial(B),$$

for every f.g. right B -module $M \neq 0$.

- (3) If B is GKdim-Cohen-Macaulay, we just say it is *Cohen-Macaulay* or *CM*.
 (4) Let ∂ be an exact dimension function and $\partial(B) = d \in \mathbb{N}$. Let s be a nonnegative integer. We say B is ∂ -CM(s) if, for every f.g right B -module M ,

$$\partial(M) \leq d - s \iff j(M) \geq s.$$

The CM property together with the Artin-Schelter regularity and the Auslander property (another homological property) has been studied in the noncommutative setting in [ASZ1, ASZ2, SZ, YZ, Zh].

Remark 1.5. In parts (4,5), assume that \mathbb{k} is a field.

- (1) The dimension function ∂ is exact on ∂ -CM algebras [ASZ2, p. 3].
- (2) GKdim is exact on filtered algebras such that the associated graded algebras are locally finite and noetherian [KL, Theorem 6.14].
- (3) GKdim satisfies $\gamma_{n,0}$ for all n . Namely, for any $N \in \text{mod}_n A$, we have $\text{GKdim } N \otimes_A M \leq n$, assuming ${}_A M_B$ is f.g. in both sides. This follows from [KL, Proposition 5.6] by a slight modification of the proof.
- (4) For every f.g. module N , it is well-known that $\text{GKdim } N = 0$ if and only if N is finite dimensional over \mathbb{k} . Using this fact, one can easily check that $\text{GKdim } \text{Tor}_i^A(N, M) = 0$ if $\text{GKdim } N = 0$ and ${}_A M_B$ is f.g. in both sides. This means that GKdim satisfies $\gamma_{0,i}$ for all i .

- (5) Take $\partial = \text{GKdim}$. As a consequence of part (4) and Lemma 1.3(2), the functor

$$- \otimes_{\mathcal{A}} \mathcal{M} : \text{QMod}_0 A \longrightarrow \text{QMod}_0 B$$

is always well-defined.

Let $B = \bigoplus_{n \geq 0} B_n$ be a noetherian graded algebra. In this paper, a graded algebra means an \mathbb{N} -graded and a graded module means \mathbb{Z} -graded. We call B *locally finite* if each B_n has finite rank over \mathbb{k} . We write $\text{Gr } B$ for the category of right graded B -modules, and $\text{gr } B$ for the subcategory of f.g. right graded B -modules. Let ∂ be an exact dimension function on graded right B -modules. Similar to the nongraded case, we write $\text{Gr}_n B$ (respectively, $\text{gr}_n B$) for the subcategory of $\text{Gr } B$ (respectively, $\text{gr } B$) consisting of objects with ∂ -dimension not larger than n . Similar to the definitions given in the introduction, we have quotient categories

$$\text{QGr}_n B := \frac{\text{Gr } B}{\text{Gr}_n B} \quad \text{and} \quad \text{qgr}_n B := \frac{\text{gr } B}{\text{gr}_n B}.$$

We also use $\pi : \text{Gr } B \rightarrow \text{QGr}_n B$ for the natural projection functor, and for $M \in \text{Gr } B$, we write $\mathcal{M} := \pi(M)$.

If M is a f.g. graded right module over a noetherian locally finite graded algebra B , then its GK-dimension can be computed by [Zh, (E7)]

$$(E1.5.1) \quad \text{GKdim } M = \overline{\lim}_{k \rightarrow \infty} \log_k \sum_{j \leq k} \dim(M_j).$$

The i th derived functor of the $\underline{\text{Hom}}_A(M, N)$ in the graded category (respectively, the graded $M \otimes_A N$) is denoted by $\underline{\text{Ext}}_A^i(M, N)$ (respectively, $\underline{\text{Tor}}_i^A(M, N)$).

Lemma 1.6. *Let A and B be noetherian locally finite graded algebras. Then GKdim on graded modules satisfies $\gamma_{n,i}$ for all n and i .*

Proof. Let M be a graded A - B -bimodule such that M is f.g. on both sides. We need to show that, for every f.g. graded right A -module N with $\text{GKdim } N \leq n$, we have $\text{GKdim } \underline{\text{Tor}}_i^A(N, M) \leq n$ for all $i \geq 0$.

Since N is f.g., $\underline{\text{Tor}}_i^A(N, M)$ is a graded f.g. B -module for each $i \geq 0$. Let

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a graded free resolution of the left graded A -module M . Since ${}_A M$ is f.g., we can choose each F_i to be f.g.. By definition, $\underline{\text{Tor}}_i^A(N, M)$ (forgetting the right B -module structure) is a subquotient of $N \otimes_A F_i$. Temporarily write $T := \underline{\text{Tor}}_i^A(N, M)$ and $\overline{N} := N \otimes_A F_i$. Since T is f.g. as a right graded B -module and, as a graded \mathbb{k} -module, T is a subquotient of \overline{N} . By (E1.5.1), we have

$$\begin{aligned} \text{GKdim } T &= \overline{\lim}_{k \rightarrow \infty} \log_k \sum_{j \leq k} \dim(T_j) \\ &\leq \overline{\lim}_{k \rightarrow \infty} \log_k \sum_{j \leq k} \dim(\overline{N}_j) \\ &= \overline{\lim}_{k \rightarrow \infty} \log_k \sum_{j \leq k} \dim(N_j) \\ &\leq \text{GKdim } N = n. \end{aligned}$$

□

Lemma 1.7. *Let A and B be noetherian locally finite graded algebras. Consider $\partial := \text{GKdim}$. For any graded A - B -bimodule M that is f.g. on both sides, a graded version of Lemma 1.3 holds. Namely, we have functors:*

$$- \otimes_A \mathcal{M} : \text{QGr}_n A \longrightarrow \text{QGr}_n B, \quad \text{and} \quad - \otimes_A \mathcal{M} : \text{qgr}_n A \longrightarrow \text{qgr}_n B.$$

Proof. By Lemma 1.6, ∂ satisfies $\gamma_{n,i}(M)$ for all n and i and all M . The assertion follows from a graded version of the proof of Lemma 1.3. \square

To conclude this section we recall and introduce a couple of more definitions.

The notion of Artin-Schelter Gorensteinness was generalized to the nonconnected graded case by several authors. We are going to use the following version. Let A be a locally finite graded algebra such that A_0 is semisimple.

Definition 1.8. [MM, RRZ] Let \mathbb{k} be a field. A locally finite graded algebra A with A_0 being semisimple is called *Artin-Schelter Gorenstein* (or *AS Gorenstein*) if the following conditions hold:

- (a) A has finite injective dimension d on both sides,
- (b) $\underline{\text{Ext}}_A^i(A_0, A) = \underline{\text{Ext}}_{A^{op}}^i(A_0, A) = 0$ for all $i \neq d$ where $A_0 = A/A_{\geq 1}$, and
- (c) $\underline{\text{Ext}}_A^d(A_0, A) \cong A_0(l)$ and $\underline{\text{Ext}}_{A^{op}}^d(A_0, A) \cong A_0(l)$ for some integer l .

If moreover

- (d) A has finite global dimension, and
- (e) A has finite GK-dimension,

then A is called *Artin-Schelter regular* (or *AS regular*).

In [MM], the AS Gorensteinness does not require that A_0 is semisimple. In the introduction, R is connected graded AS regular.

When A is noetherian, connected graded and PI of finite global dimension, then A is AS regular and CM [SZ, Theorem 1.1]. It is conjectured that every noetherian connected graded AS regular algebra is CM.

Definition 1.9. Let R be a subalgebra of B and let ∂ be a dimension function defined on right R -modules and right B -modules.

- (1) We say ∂ is *weakly B/R -hereditary* if $\partial(M_R) \leq \partial(M_B)$ for every f.g. right B -module M .
- (2) Suppose B_R is f.g.. We say ∂ is *B/R -hereditary* if for every f.g. right B -module M , $\partial(M_R) = \partial(M_B)$.

Lemma 1.10. *Let R be a subalgebra of B .*

- (1) *GKdim is always weakly B/R -hereditary.*
- (2) *Suppose that $R \subseteq B$ are noetherian and locally finite graded algebras such that B_R is f.g. Then GKdim is B/R -hereditary on graded right modules.*

Proof. (1) This follows from Definition 1.1(2).

(2) Let M be a f.g. graded right B -module. Since B_R is f.g., M is a f.g. R -module. By (E1.5.1), $\text{GKdim } M$ is only dependent on the Hilbert series of M , not the module structure. Therefore $\text{GKdim } M_B = \text{GKdim } M_R$. \square

2. A MORITA TYPE THEOREM

The main result of this section is Theorem 2.4. We consider the following hypotheses to be used in different situations.

- Hypothesis 2.1.** (1) A and B are noetherian algebras.
 (2) Let e be an idempotent in B and $A = eBe$.
 (3) ∂ is a dimension function defined on right A -modules and B -modules and $\partial(B) =: d \geq 2$.
 (4) B is a ∂ -CM(2) algebra.
 (5) eB and Be are f.g. A -modules.
 (6) ∂ is exact on right A -modules.
 (7) For every f.g. right B -module N , $\partial((Ne)_A) (= \partial(N \otimes_B Be)) \leq \partial(N_B)$.

Under Hypothesis 2.1(2), Be is a right A -module and there is a natural algebra morphism

$$(E2.1.1) \quad \varphi : B \rightarrow \text{End}_A(Be), \quad \varphi(b)(b'e) = bb'e$$

induced by the left multiplication. In this section we will investigate when this morphism φ is an isomorphism. A special case of this situation is the Auslander theorem [Au1, Proposition 3.4].

We need use some weaker versions of CM condition. Let ∂ be a dimension function and α be an integer.

- (a) We say an algebra B is ∂ -CM^l(α) if

$$\partial(M) \leq d - s \implies j(M) \geq s.$$

- (b) We say an algebra B is ∂ -CM^r(α) if

$$\partial(M) \leq d - s \iff j(M) \geq s.$$

Then B is ∂ -CM(α) if and only if it is both ∂ -CM^l(α) and ∂ -CM^r(α).

Lemma 2.2. *Let (A, B) satisfy Hypothesis 2.1(1,2,3) and α be an integer such that $0 < \alpha \leq d$. Suppose that B is ∂ -CM^l(α). Let N be a f.g. right A -module. If $\partial(\text{Tor}_i^A(N, eB)) \leq d - \alpha$ for all $i \leq \alpha - 1$, then $\text{Ext}_A^i(N, Be) = 0$ for all $i \leq \alpha - 1$.*

Proof. We have canonical isomorphisms induced by the adjointness

$$\text{Hom}_B(N \otimes_A eB, B) \cong \text{Hom}_A(N, \text{Hom}_B(eB, B)) \cong \text{Hom}_A(N, Be).$$

From these isomorphisms, we obtain a spectral sequence [Ro, Theorem 11.54]

$$(E2.2.1) \quad E_2^{pq} = \text{Ext}_B^p(\text{Tor}_q^A(N, eB), B) \implies \text{Ext}_A^{p+q}(N, Be).$$

By hypothesis, B is ∂ -CM^l(α), we have $\text{Ext}_B^p(\text{Tor}_q^A(N, eB), B) = 0$ for all $q \leq \alpha - 1$ and $p \leq \alpha - 1$ since $\partial(\text{Tor}_i^A(N, eB)) \leq d - \alpha$ for all $i \leq \alpha - 1$. By (E2.2.1), we obtain $\text{Ext}_A^i(N, Be) = 0$ for all $i \leq \alpha - 1$. \square

In the next lemma it is not necessary to assume $d \geq 2$.

Lemma 2.3. *Let (A, B) satisfy Hypothesis 2.1(1-3), (5-7) and α be a positive integer. Suppose that*

$$(E2.3.1) \quad \partial \text{ satisfies } \gamma_{d-\alpha, 1}(eB).$$

If $\partial(B/(BeB)) \leq d - \alpha$, then the functor $- \otimes_B Be : \text{qmod}_{d-\alpha} B \longrightarrow \text{qmod}_{d-\alpha} A$ is an equivalence.

Proof. By hypothesis (E2.3.1), ∂ satisfies $\gamma_{d-\alpha,1}(eB)$. By Lemma 1.3, we have the induced functor $-\otimes_A e\mathcal{B} : \text{qmod}_{d-\alpha} A \longrightarrow \text{qmod}_{d-\alpha} B$. Since Be is projective as a left B -module, the hypothesis of Lemma 1.3 follows from Hypothesis 2.1(7) with $M = Be$ and A and B are switched. By Lemma 1.3, we have the induced functor $-\otimes_B \mathcal{B}e : \text{qmod}_{d-\alpha} B \longrightarrow \text{qmod}_{d-\alpha} A$. For any $N \in \text{mod } A$, we have natural isomorphisms in $\text{qmod}_{d-\alpha} A$:

$$\mathcal{N} \otimes_A e\mathcal{B} \otimes_B \mathcal{B}e \cong \pi(N \otimes_A eB \otimes_B \mathcal{B}e) \cong \pi(N) = \mathcal{N}.$$

Taking $M \in \text{mod } B$, we have natural isomorphisms in $\text{qmod}_{d-\alpha} B$:

$$\mathcal{M} \otimes_B \mathcal{B}e \otimes_A e\mathcal{B} \cong \pi(M \otimes_B Be \otimes_A eB).$$

We claim that $\pi(M \otimes_B Be \otimes_A eB) = \pi(M)$. Let $f : Be \otimes_A eB \rightarrow B$ be induced by the multiplication in B and factor f as $f = hg$ with h a monomorphism and g an epimorphism. Then we have two short exact sequences as follows,

$$(E2.3.2) \quad 0 \rightarrow K \rightarrow Be \otimes_A eB \xrightarrow{g} BeB \rightarrow 0,$$

$$(E2.3.3) \quad 0 \rightarrow BeB \xrightarrow{h} B \rightarrow B/BeB \rightarrow 0.$$

Applying the functor $-\otimes_B Be$ to (E2.3.2), we obtain the exact sequence

$$0 \rightarrow Ke \rightarrow Be \otimes_A eBe \rightarrow Be \rightarrow 0,$$

which results that $Ke = 0$. Similarly, $eK = 0$. Since Be is a f.g. right A -module, $Be \otimes_A eB$ is a f.g. right B -module. Hence K is a f.g. B/BeB -bimodule. Since $\partial(B/BeB) \leq d - \alpha$, we have $\partial(K) \leq d - \alpha$. Consider the exact sequence

$$M \otimes_B K \rightarrow M \otimes_B Be \otimes_A eB \rightarrow M \otimes_B BeB \rightarrow 0.$$

Since $M \otimes_B K$ is a f.g. B/BeB -module, $\partial(M \otimes_B K) \leq d - \alpha$. Now we have $\pi(M \otimes_B Be \otimes_A eB) \cong \pi(M \otimes_B BeB)$ in $\text{qmod}_{d-\alpha} B$. From the exact sequence (E2.3.3), we obtain an exact sequence

$$\text{Tor}_1^B(M, B/BeB) \rightarrow M \otimes_B BeB \rightarrow M \rightarrow M \otimes_B B/BeB \rightarrow 0.$$

Since the ∂ -dimension of B/BeB is not larger than $d - \alpha$, neither are the ∂ -dimensions of $M \otimes_B B/BeB$ and $\text{Tor}_1^B(M, B/BeB)$. Hence we have natural isomorphisms in $\text{qmod}_{d-\alpha} B$: $\pi(M \otimes_B BeB) \cong \pi(M)$. Summarizing, we obtain

$$\mathcal{M} \otimes_B \mathcal{B}e \otimes_A e\mathcal{B} \cong \pi(M \otimes_B Be \otimes_A eB) \cong \pi(M) = \mathcal{M}$$

as desired. \square

A special case of Lemma 2.3 is the following well-known statement: if e is an idempotent in B such that $BeB = B$, then B is Morita equivalent to $A := eBe$. Most of Hypothesis 2.1 is not necessary in this special case. Therefore Lemma 2.3 can be viewed as a generalization of Morita theorem. Specializing the last statement to a semisimple H -action on an algebra A , if $e := 1 \# \int$ in $B := A \# H$ is such that $BeB = B$, then A^H is Morita equivalent to $A \# H$, see [Mon, Corollary 4.5.4].

The main result of this section is the following Morita type theorem.

Theorem 2.4. *Let (A, B) satisfy Hypothesis 2.1(1-7). Suppose*

$$(E2.4.1) \quad \partial \text{ satisfies } \gamma_{d-2,1}(eB).$$

Then the following statements are equivalent.

- (i) *The functor $-\otimes_B \mathcal{B}e : \text{qmod}_{d-2} B \longrightarrow \text{qmod}_{d-2} A$ is an equivalence.*

- (ii) *The natural map φ of (E2.1.1) is an isomorphism of algebras.*
- (iii) $\partial(B/(BeB)) \leq d-2$.
- (iv) $j(B/BeB) \geq 2$.

Proof. (i) \implies (ii). Let $F = - \otimes_B \mathcal{B}e$. Since F is an equivalence, we obtain isomorphisms of algebras:

$$(E2.4.2) \quad \text{End}_{\text{qmod}_{d-2} B} \mathcal{B} \cong \text{End}_{\text{qmod}_{d-2} A} F(\mathcal{B}) = \text{End}_{\text{qmod}_{d-2} A} \pi(Be).$$

By assumption, B is $\partial\text{-CM}^l(2)$. Hence, for $M \in \text{mod}_{d-2} B$,

$$\text{Hom}_B(M, B) = 0 = \text{Ext}_B^1(M, B).$$

As a consequence, ${}_B B$ does not contain any nonzero submodule of ∂ -dimension at most $d-2$. Therefore, by definition,

$$\text{End}_{\text{qmod}_{d-2} B} \mathcal{B} = \varinjlim \text{Hom}_B(K, B),$$

where the limit runs over all the right submodule modules $K \subseteq B$ such that $\partial(B/K) \leq d-2$. For such a submodule K , we have

$$\text{Hom}_B(B/K, B) = 0 = \text{Ext}_B^1(B/K, B).$$

Hence, a long exact sequence implies that $\text{Hom}_B(K, B) = \text{Hom}_B(B, B) = B$. Thus, we have $\text{End}_{\text{qmod}_{d-2} B} \mathcal{B} \cong B$.

Let $N \in \text{mod}_{d-2} A$. By Lemma 1.3(1) and (E2.4.1), the hypothesis of Lemma 2.2 concerning $\partial(\text{Tor}_i^A(N, eB))$ holds for $\alpha = 2$. Hence, by Lemma 2.2,

$$(E2.4.3) \quad \text{Hom}_A(N, Be) = 0 = \text{Ext}_A^1(N, Be)$$

for all $N \in \text{mod}_{d-2} A$. In particular, Be does not have any nonzero right A -submodule of ∂ -dimension $\leq d-2$. Using this and (E2.4.3), we have

$$\text{Hom}_{\text{qmod}_{d-2} A}(\mathcal{B}e, \mathcal{B}e) = \varinjlim \text{Hom}_A(K, Be) \cong \text{Hom}_A(Be, Be),$$

where the limit runs over all the submodules $K \subseteq Be$ such that $\partial(Be/K) \leq d-2$. Combining with (E2.4.2), one sees that $B \cong \text{Hom}_A(Be, Be)$, and the isomorphism is the natural morphism given in statement (ii).

(ii) \implies (iv). Let $f : Be \otimes_A eB \longrightarrow B$ be the B -bimodule morphism defined by $f(be \otimes_A eb') = beb'$, which was used in the proof of Lemma 2.3. Consider the following composition of morphisms:

$$\begin{aligned} B = \text{Hom}_B(B, B) &\xrightarrow{\text{Hom}_B(f, B)} \text{Hom}_B(Be \otimes_A eB, B) \\ &\xrightarrow{\phi} \text{Hom}_A(Be, \text{Hom}_B(eB, B)) \xrightarrow{\alpha} \text{Hom}_A(Be, Be), \end{aligned}$$

where ϕ is the natural isomorphism defined by the adjointness, and α is the isomorphism induced by the isomorphism $\text{Hom}_B(eB, B) \cong Be$. One sees that the composition above is equal to the natural morphism

$$\varphi : B \longrightarrow \text{End}_A(Be), \quad \varphi(b)(b'e) = bb'e.$$

By (ii), it is an isomorphism. Hence the morphism

$$\text{Hom}_B(f, B) : \text{Hom}_B(B, B) \longrightarrow \text{Hom}_B(Be \otimes_A eB, B)$$

is an isomorphism. Let us factor the morphism f as follows:

$$\begin{array}{ccc} Be \otimes_A eB & \xrightarrow{f} & B \\ & \searrow g & \nearrow h \\ & C, & \end{array}$$

where $C = \text{im} f = BeB$, g is the epimorphism induced by f and h is the inclusion map. Applying the functor $\text{Hom}_B(-, B)$ to the diagram above, we have $\text{Hom}_B(f, B) = \text{Hom}_B(g, B) \circ \text{Hom}_B(h, B)$. Since g is an epimorphism, it follows that $\text{Hom}_B(g, B)$ is a monomorphism, and hence it has to be an isomorphism, which in turn implies that $\text{Hom}_B(h, B)$ is an isomorphism. From the exact sequence

$$0 \longrightarrow C \xrightarrow{h} B \longrightarrow B/BeB \longrightarrow 0,$$

we obtain the following exact sequence

$$0 \longrightarrow \text{Hom}_B(B/BeB, B) \longrightarrow \text{Hom}_B(B, B) \xrightarrow{\text{Hom}_B(h, B)} \text{Hom}_B(C, B) \longrightarrow \text{Ext}_B^1(B/BeB, B) \longrightarrow 0.$$

Since $\text{Hom}_B(h, B)$ is an isomorphism, we obtain $\text{Hom}_B(B/BeB, B) = 0$ and $\text{Ext}_B^1(B/BeB, B) = 0$. Hence $j(B/(BeB)) \geq 2$.

(iv) \implies (iii) This follows from $\partial\text{-CM}^r(2)$.

(iii) \implies (i). This is a special case of Lemma 2.3 when $\alpha = 2$. \square

Remark 2.5. We have the following observations concerning the proof of Theorem 2.4.

- (1) In the proof of (i) \implies (ii), only $\partial\text{-CM}^l(2)$ is used (not the entire $\partial\text{-CM}(2)$).
- (2) In the proof of (iv) \implies (iii), only $\partial\text{-CM}^r(2)$ is used (not the entire $\partial\text{-CM}(2)$). In fact, we only used the fact that $j(B/BeB) \geq 2$ implies that $\partial(B/BeB) \leq d - 2$.
- (3) In the proof of (iii) \implies (i), $\partial\text{-CM}(2)$ is not needed.
- (4) (iii) \implies (iv) Follows from the definition of $\partial\text{-CM}(2)$.

Now let $B = \bigoplus_{i \geq 0} B_i$ be a noetherian locally finite graded algebra. Let $e \in B_0$ be an idempotent. Then $A := eBe$ is a noetherian locally finite graded algebra. Following Definition 1.4, B is said to be *graded CM* if $\text{GKdim } B = d < \infty$ and $j(M) + \text{GKdim } K = \text{GKdim } B$ for all f.g. graded right B -modules M .

Note that the GK-dimension is always exact on modules over noetherian locally finite graded algebras [Remark 1.5(2)]. By Lemma 1.6, GKdim satisfies $\gamma_{n,i}$. Note that, by (E1.5.1), GKdim satisfies Hypothesis 2.1(7) in the locally finite graded case. Therefore we obtain the following graded version of Theorem 2.4.

Theorem 2.6. *Let $B = \bigoplus_{i \geq 0} B_i$ be a noetherian locally finite graded algebra. Suppose (A, B) satisfies Hypothesis 2.1(1-5) in the graded setting with $e \in B_0$ and $\partial = \text{GKdim}$. Then the following statements are equivalent.*

- (i) *The functor $- \otimes_B Be : \text{qgr}_{d-2} B \longrightarrow \text{qgr}_{d-2} A$ is an equivalence.*
- (ii) *The natural map $\varphi : B \longrightarrow \text{End}_A(Be)$ defined by $\varphi(b)(b'e) = bb'e$ is an isomorphism of graded algebras.*
- (iii) *$\text{GKdim } B/(BeB) \leq d - 2$.*
- (iv) *$j(B/BeB) \geq 2$.*

3. HOPF ALGEBRA ACTIONS ON GRADED ALGEBRAS

In this section we consider locally finite graded algebras and let ∂ be a dimension function on locally finite graded modules. Starting from Proposition 3.3, we further assume that \mathbb{k} is a field.

Let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra that is free of finite rank over \mathbb{k} . Let R be a noetherian locally finite graded algebra. Assume that R is a graded left H -module algebra, i.e., H acts on R homogeneously. Then the smash product $R \# H$ is a noetherian locally finite graded algebra.

Suppose that H has a (left and right) integral \int such that $\varepsilon(\int) = 1$. If \mathbb{k} is a field, this is equivalent to the fact that H is semisimple. Let $e = 1 \# \int \in R \# H$. One sees that e is an idempotent of $R \# H$. Write the fixed subring of the H -action on R as

$$(E3.0.1) \quad R^H = \{r \in R \mid h \cdot r = \varepsilon(h)r, \forall h \in H\}.$$

Then R^H is a graded subalgebra of R . The following is well-known [Mon].

Lemma 3.1. *With the notation as above. The following hold.*

- (1) R^H is noetherian.
- (2) R is f.g. as a left graded R^H -module and as a right graded R^H -module.
- (3) The map $R^H \longrightarrow e(R \# H)e$, sending $r \mapsto e(r \# 1)e$, is an isomorphism of graded algebras.
- (4) The map $R \longrightarrow (R \# H)e$, sending $r \mapsto (r \# 1)e$, is an isomorphism of graded (R, R^H) -modules.
- (5) The map $R \longrightarrow e(R \# H)$, sending $r \mapsto e(r \# 1)$, is an isomorphism of graded (R^H, R) -modules.

Proof. (1) This follows by the proof of [Mon, Corollary 4.3.5].

(2) This follows by the proof of [Mon, Theorem 4.4.2].

(3,4,5) By direct computation. □

Hypothesis 3.2. (1) R is a noetherian, locally finite, graded algebra.

- (2) H is a Hopf algebra acting on R homogeneously, and H is free of finite rank over \mathbb{k} with integral \int such that $\varepsilon(\int) = 1$.
- (3) Let B be the algebra $R \# H$ with $e := 1 \# \int \in B$. Identifying R^H with eBe by Lemma 3.1(3), R with Be by Lemma 3.1(5).
- (4) Let ∂ be an exact dimension function on f.g. right graded B -modules, R -modules, and R^H -modules. This is automatic if $\partial = \text{GKdim}$.
- (5) ∂ is B/R -hereditary. This is automatic if $\partial = \text{GKdim}$ by Lemma 1.10(2).
- (6) $\partial(R) = d \geq 2$ and R is graded ∂ -CM(2)

Here is our first result in this section. For the rest of this section we assume that \mathbb{k} is a field.

Proposition 3.3. *Retain Hypothesis 3.2(1-5). If R is ∂ -CM (respectively, ∂ -CM(s), ∂ -CM ^{l} (s), ∂ -CM ^{r} (s)), then so is $B := R \# H$.*

Proof. Since B is isomorphic to $R^{\oplus \dim H}$ as a right R -module, we have

$$\partial(B_B) = \partial(B_R) = \partial(R_R) =: d.$$

Let M be a f.g. graded right B -module with ∂ -dimension i . Since ∂ is B/R -hereditary, M , viewed as a right R -module, has ∂ -dimension i , or $\partial(M_B) = \partial(M_R)$. It remains to show that $j(M_B) = j(M_R)$.

The cohomology $\underline{\text{Ext}}_B^j(M, B)$ can be computed by $\underline{\text{Ext}}_R^j(M, B)$. Since H is semisimple, by [Ne, Corollary 7.6], we have

$$(E3.3.1) \quad \underline{\text{Ext}}_B^j(M, B) \cong \underline{\text{Ext}}_R^j(M, B)^H.$$

Equation (E3.3.1) follows also from a more general result [St, Theorem 3.3] together with [We, Lemma 9.1.9].

We recall the right H -module structure on $\underline{\text{Ext}}_R^j(M, B)$ as follows. Let

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M_B \longrightarrow 0$$

be a f.g. projective resolution of M_B . Let N be a right B -module. For $j \geq 0$, $\underline{\text{Hom}}_R(P^{-j}, N)$ has a right H -module structure defined by

$$(f \cdot h)(p) = f(p(Sh_{(1)}))h_{(2)}$$

for all $p \in P^{-j}$, $f \in \underline{\text{Hom}}_R(P^{-j}, N)$ and $h \in H$, where S is the antipode of H and we use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$. The right H -action is compatible with the differential of the complex $\underline{\text{Hom}}_R(P^\bullet, N)$, from which we obtain the right H -action on $\underline{\text{Ext}}_R^j(M, N)$.

We may form another smash product $H \# R$ as follows [RRZ, Lemma 2.1]: as a vector space $H \# R = H \otimes R$, the product is defined by

$$(h \# r)(g \# r') = hg_{(2)} \# (S^{-1}g_{(1)} \cdot r)r'.$$

The algebras $R \# H$ and $H \# R$ are isomorphic through the isomorphism:

$$\varphi : R \# H \longrightarrow H \# R,$$

defined by

$$r \# h \mapsto h_{(2)} \# S^{-1}h_{(1)} \cdot r.$$

For all $j \geq 0$, we have the following morphism

$$\Psi : \underline{\text{Hom}}_R(P^{-j}, R \# H) \longrightarrow \underline{\text{Hom}}_R(P^{-j}, H \# R),$$

defined by

$$\Psi(f) = \varphi \circ f,$$

and

$$\Phi : \underline{\text{Hom}}_R(P^{-j}, R) \otimes H \longrightarrow \underline{\text{Hom}}_R(P^{-j}, H \# R),$$

defined by

$$\Phi(f \otimes h)(p) = h \# f(p).$$

One may check that both Ψ and Φ are isomorphisms of vector spaces and are compatible with the differentials. Through the isomorphism φ , we view $H \# R$ as a right $R \# H$ -module. Note that R can be viewed as a right $R \# H$ -module via the action

$$r \cdot (r' \# h) = S^{-1}h(rr')$$

for $r, r' \in R$ and $h \in H$. Hence both $\underline{\text{Hom}}_R(P^{-j}, H \# R)$ and $\underline{\text{Hom}}_R(P^{-j}, R)$ are right H -modules. Now $\underline{\text{Hom}}_R(P^{-j}, R) \otimes H$ is also a right H -module by the usual H -action defined by

$$(f \otimes g) \cdot h = f \cdot h_{(1)} \otimes gh_{(2)}$$

for $f \in \underline{\text{Hom}}_R(P^{-j}, R)$ and $g, h \in H$. Since φ is an isomorphism of algebras, Ψ is an isomorphism of right H -modules. We next check that Φ is a right H -module morphism, and hence it is indeed an isomorphism of right H -modules. For $f \in \underline{\text{Hom}}_R(P^{-j}, R)$, $h, g \in H$ and $p \in P^{-j}$, we have

$$\begin{aligned} [\Phi(f \otimes h) \cdot g](p) &= [(f \otimes h)(pSg_{(1)})]g_{(2)} \\ &= [h\#f(pSg_{(1)})]g_{(2)} \\ &= hg_{(3)}\#S^{-1}g_{(2)}f(pSg_{(1)}), \\ \Phi((f \otimes h)g)(p) &= \Phi(f \cdot g_{(1)} \otimes hg_{(2)})(p) \\ &= hg_{(2)} \otimes (f \cdot g_{(1)})(p) \\ &= hg_{(3)} \otimes f(pSg_{(1)}) \cdot g_{(2)} \\ &= hg_{(3)} \otimes S^{-1}g_{(2)}f(pSg_{(1)}). \end{aligned}$$

Hence Φ is a right H -module morphism. Therefore the composition $\Phi^{-1}\Psi$ induces an isomorphism of complexes of right H -modules

$$\underline{\text{Hom}}_R(P^\bullet, R\#H) \longrightarrow \underline{\text{Hom}}_R(P^\bullet, R) \otimes H.$$

Applying $(-)^H$, we obtain an isomorphism of complexes

$$\underline{\text{Hom}}_R(P^\bullet, R\#H)^H \cong (\underline{\text{Hom}}_R(P^\bullet, R) \otimes H)^H.$$

Since $(-)^H$ is an exact functor, taking cohomology $H^j(-)$ and taking the invariants $(-)^H$ commutes. Therefore we have

$$\underline{\text{Ext}}_R^j(M, R\#H)^H \cong (\underline{\text{Ext}}_R^j(M, R) \otimes H)^H.$$

Combining with the isomorphism (E3.3.1), we finally obtain

$$(E3.3.2) \quad \underline{\text{Ext}}_B^j(M, B) \cong (\underline{\text{Ext}}_R^j(M, R) \otimes H)^H.$$

By (E3.3.2), it is clear that $j(M_B) \geq j(M_R)$. Assume that x_1, \dots, x_n is a \mathbb{k} -basis of H . Then we may write the coproduct of \int as $\Delta(\int) = \sum_{s=1}^n y_s \otimes x_s$. Hence we see that y_1, \dots, y_n generates H [Sw, Proposition 2.6]. If $\underline{\text{Ext}}_R^j(M, R) \neq 0$, there is a nonzero element $\alpha \in \underline{\text{Ext}}_R^j(M, R)$ such that $\alpha y_s \neq 0$ for some s . Now $\sum_{s=1}^n \alpha y_s \otimes x_s \neq 0$, which is in $(\underline{\text{Ext}}_R^j(M, R) \otimes H)^H$. Therefore $\underline{\text{Ext}}_B^j(M, B) \neq 0$. This means that $j(M_B) \leq j(M_R)$. Therefore $j(M_B) = j(M_R)$. \square

Combine Lemmas 2.2, 1.6 and 3.1 and Proposition 3.3, we have the following consequences.

Proposition 3.4. *Retain Hypothesis 3.2(1,2,3) with $\partial = \text{GKdim}$ and assume that \mathbb{k} is a field and that R is $\text{CM}(\alpha)$ for a positive integer α . Then $\underline{\text{Ext}}_{R^H}^i(N, R) = 0$ for all $N \in \text{gr}_{d-\alpha} R^H$ and all $i \leq \alpha - 1$.*

Proof. Let $(A, B) = (R^H, R\#H)$. Then Hypothesis 2.1(1,2,3) can be verified easily. By Proposition 3.3, B is $\text{CM}(\alpha)$. The assertion follows from Lemmas 1.6 and 2.2. \square

Assisted by Proposition 3.3, Theorem 2.6 now reads in the following form. Recall that the pertinency of H -action on R [Definition 0.1] is defined to be

$$\text{p}(R, H) = \text{GKdim } R - \text{GKdim}(R\#H)/(e)$$

where $e = 1\#\int$.

Theorem 3.5. *Retain Hypothesis 3.2(1-3) with $\partial = \text{GKdim}$. Assume that R is CM(2) with $\text{GKdim } R \geq 2$. Set $(A, B) = (R^H, R\#H)$. Then the following are equivalent.*

- (i) *The functor $-\otimes_B \mathcal{R} : \text{qgr}_{d-2} B \longrightarrow \text{qgr}_{d-2} A$ is an equivalence.*
- (ii) *The natural map $\varphi : B \longrightarrow \text{End}_A(R)$ is an isomorphism of algebras.*
- (iii) *$\text{p}(R, H) \geq 2$.*
- (iv) *H -action is homologically small in the sense of Definition 0.2(2).*

Proof. This is an immediate consequence of Theorem 2.6 after matching up with notations and verifying the hypotheses by Proposition 3.3. Note that (iii) and (iv) are equivalent by CM(2). \square

Remark 3.6. (1) Theorem 0.3 is a special case of Theorem 3.5.

(2) A version of Theorem 3.5 holds when R is ungraded (or R is not locally finite) as long as

- (i) the GK-dimension on right R^H -modules are exact,
- (ii) the GK-dimension is $R\#H/R^H$ -hereditary,
- (iii) $\gamma_{n,1}(eB)$ holds for GKdim .

(3) If $R = \mathbb{C}[x_1, \dots, x_d]$ is the polynomial algebra, and G a finite small subgroup of $\text{GL}_d(\mathbb{C})$, the Auslander theorem says the natural map $R\#\mathbb{k}G \longrightarrow \text{End}_{R^G}(R)$ as in Theorem 3.5 is an isomorphism of graded algebras [Au1, Yo, IT]. By Theorem 3.5, $\text{p}(R, G) \geq 2$. See comments in [BHZ, Section 7].

(4) If R/R^H is an H^* -dense Galois extension in the sense of [HVZ2], then Theorem 3.5 partially recovers [HVZ2, Theorem 3.8].

(5) Theorem 3.5 also partially recovers [MU, Theorem 3.7].

Next we prove Theorem 0.6.

Theorem 3.7. *Retain Hypothesis 3.2(1,2,3). Let α be an integer no more than $\text{p}(R, H)$. Then there is an equivalence of categories*

$$\text{qgr}_{d-\alpha} R^H \cong \text{qgr}_{d-\alpha} R\#H.$$

Proof. By Lemma 1.6, hypothesis (E2.3.1) holds for $\partial = \text{GKdim}$ on graded modules. Hypothesis 2.1(1-3)(5-7) can be checked one by one. Therefore the assertion follows by Lemma 2.3. \square

Corollary 3.8. *Retain Hypothesis 3.2(1,2,3). Let R be an AS regular algebra. If $\text{p}(R, H) = \text{GKdim } R$, then R^H has graded isolated singularities.*

Proof. By Theorem 3.7, $\text{qgr}_{d-\alpha} R^H \cong \text{qgr}_{d-\alpha} R\#H$ where the latter has finite global dimension. The assertion follows by the definition, see [Ue, Definition 2.2]. \square

For the rest of this section we make some observations the condition to ensure that $\text{p}(R, H) \geq 1$. We recall a definition below.

Definition 3.9. A left H -module M is called *inner faithful* if $IM \neq 0$ for every nonzero Hopf ideal I of H . We say that an H -action on A is inner faithful, if A is an inner faithful left H -module.

Lemma 3.10. *Retain Hypothesis 3.2(1-5) with $\partial = \text{GKdim}$. Assume that \mathbb{k} is a field. Suppose that R is a domain.*

- (1) [SkV, Theorem 0.5(i)] $B := R\#H$ is semiprime.

- (2) [BCF, Corollary 3.4] *B is prime if and only if the canonical algebra homomorphism $\phi : B \rightarrow \text{End}_{R^\#}(R)$ is injective.*
- (3) *If the map ϕ in part (2) is injective, then the H -action on R is inner faithful. As a consequence, if B is prime, then the H -action is inner faithful.*
- (4) *B is prime if and only if $\mathfrak{p}(R, H) \geq 1$.*

Proof. (1) This is a special case of [SkV, Theorem 0.5(i)].

(2) Condition [BCF, Lemma 3.3(ii)] can be checked by using GK-dimension. The assertion follows from [BCF, Corollary 3.4].

(3) If the H -action on R is not inner faithful, there is a Hopf ideal $0 \neq R \subseteq H$ such that $JR = 0$. Thus $\phi(1 \# j) = 0$ for all $j \in J$. The assertion follows. The consequence follows from part (2).

(4) First we suppose that $\mathfrak{p}(R, H) \geq 1$. By part (1), B is semiprime. Since B is a free module over R , it is d -pure with respect to the GK-dimension in the sense of [ASZ2, Definition 0.3]. Hypothesis (E2.3.1) can be checked for $\alpha = 1$ when A is a domain. By Lemma 2.3, $\text{qmod}_{d-1} B \cong \text{qmod}_{d-1} A$. Since A is an Ore domain, $\text{qmod}_{d-1} A$ is equivalent to a module category over the quotient division ring $Q(A)$ of A . Therefore, $\text{qmod}_{d-1} B$ is equivalent to a module category over a division ring. Since B is GKdim-pure, $\text{qmod}_{d-1} B \cong \text{mod } Q(B)$ where $Q(B)$ is the semisimple artinian quotient ring of B . This implies that $Q(B)$ is simple and B is prime.

The converse follows from the fact $I = (e) \subseteq B$ is a nonzero ideal and B is prime. \square

Lemma 3.10 were suggested by Ellen Kirkman. The authors thank her for sharing her ideas.

4. BGG CORRESPONDENCE

From now on let \mathbb{k} be a field. In this section, we study the quotient category $\text{qgr}_{d-\alpha} R^H$ as appeared in the last section. If R is noetherian, connected graded, AS regular and if the homological determinant of the H -action on R is trivial, then R^H is AS Gorenstein [KKZ1, Theorem 0.1]. In this case, the quotient category $\text{qgr}_{d-2} R \# H$, and hence, under the setting of Theorem 3.5 (assuming $\mathfrak{p}(R, H) \geq 2$), the quotient category $\text{qgr}_{d-2} R^H$, is of finite global dimension, while the fixed subring R^H has infinite global dimension [CKWZ1, Theorem 0.6]. Hence it is relatively easier to understand the category $\text{qgr}_{d-2} R \# H$ than to do the category $\text{qgr}_{d-2} R^H$ directly.

We are now working over Koszul AS regular algebras. Let $B = \bigoplus_{n \geq 0} B_n$ be a noetherian locally finite graded algebra such that B_0 is a semisimple algebra. Recall that a right graded B -module is called a *Koszul module* [Pr, BGS] if M has a linear graded projective resolution:

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0,$$

where P^{-n} are graded projective modules generated in degree n for all $n \geq 0$. If the right graded B -module $B_0 := B/B_{\geq 1}$ is a Koszul module, then B is called a *Koszul algebra*.

Let B be a graded algebra and let

$$(E4.0.1) \quad E = E(B) := \bigoplus_{n \geq 0} \underline{\text{Ext}}_B^n(B_0, B_0)$$

be the Yoneda algebra of B . By [BGS, Proposition 2.9.1 and Theorem 2.10.1], if B is a Koszul algebra, then the Yoneda algebra E is also Koszul.

Let B be a locally finite graded algebra such that B_0 is semisimple. Let $\{S_1, \dots, S_w\}$ be the complete set of non-isomorphic graded right B -module concentrated in degree 0, and let Q_i be the projective cover of S_i , for $i = 1, \dots, w$. Define

$$(E4.0.2) \quad B' = \underline{\text{Hom}}_B(\bigoplus_{i=1}^w Q_i, \bigoplus_{i=1}^w Q_i).$$

The following lemma is well-known.

Lemma 4.1. *Retain the notations as above and assume that B_0 is semisimple.*

- (1) *B' is graded Morita equivalent to B . Namely, there is an equivalence of graded module categories*

$$\text{Gr } B' \cong \text{Gr } B,$$

which induces an equivalence of triangulated categories

$$D(\text{Gr } B') \cong D(\text{Gr } B).$$

- (2) *$E(B)$ is graded Morita equivalent to $E(B')$.*
 (3) *B is noetherian (respectively, AS Gorenstein, AS regular, Koszul) if and only if so is B' .*
 (4) *If \mathbb{k} is algebraically closed, then B'_0 is a finite direct sum of \mathbb{k} .*

For the rest of this section, we assume that \mathbb{k} is algebraic closed. Then B'_0 is a finite direct sum of \mathbb{k} [Lemma 4.1(4)]. In this case B' is called the *basic algebra* of B . If R is as in Theorem 3.5 and assume that R is a connected graded Koszul AS regular algebra, then $R\#H$ is also a Koszul AS-regular algebra [HVZ1, Corollary 2.9]. Since the graded Morita equivalence of algebras preserves almost all algebraic invariants [Lemma 4.1], we will study the basic algebra of $R\#H$ instead. In general, one can replace B by its basic algebra B' if necessary.

From now on, $B = B_0 \oplus B_1 \oplus \dots$ is a noetherian locally finite graded algebra such that B_0 is a finite direct sum of \mathbb{k} . The following lemma is needed.

Lemma 4.2. [Ma, Theorem 5.1] *Let B be a Koszul algebra. Then B is AS regular if and only if $E := \bigoplus_{n \geq 0} \underline{\text{Ext}}_B^n(B_0, B_0)$ is a graded Frobenius algebra.*

Let B be a Koszul algebra. We may construct a differential bigraded algebra \tilde{B} by setting $\tilde{B}_j^0 = B_j$ and $\tilde{B}_j^i = 0$ for all $i \neq 0$ with zero differential. We may construct a differential bigraded algebra \tilde{E} associated to E (E4.0.1) by setting $\tilde{E}_{-i}^i = E_i$ and $\tilde{E}_j^i = 0$ for all $i \neq -j$ with zero differential. A differential bigraded \tilde{B} -module M is a bigraded \tilde{B} -module $M = \bigoplus_{i,j \in \mathbb{Z}} M_j^i$ together with a differential d such that $d(M_j^i) \subseteq M_j^{i+1}$ for all $i, j \in \mathbb{Z}$. Note that the derived category of differential bigraded right \tilde{B} -modules (respectively, \tilde{E} -modules) is equivalent to the derived category of right graded B -modules (respectively, E -modules). Then the general Koszul duality of bigraded differential graded algebras induces following duality between the derived category of B and of E [HW, Remark 4.1], which is a slightly different from the equivalence obtained in [BGS]. Note that Smith-Van den Bergh stated a similar result in [SmV].

Proposition 4.3. [SmV, Section 2.4] *Let B be a Koszul AS regular algebra. We have a duality of triangulated categories:*

$$K : D^b(\text{gr } B) \longrightarrow D^b(\text{gr } E),$$

where $K(M) = G(\mathrm{RHom}_B(M, B_0))$ for $M \in D^b(\mathrm{gr} B)$, and G is the regrading functor defined by $G(X)_s^n = X_{-s}^{n+s}$ for any complex of graded modules X^\cdot .

Note that the functor K commutes with the shift functors in the following way

$$K(X^\cdot[1]) = K(X^\cdot)[-1] \text{ and } K(X^\cdot(1)) = K(X^\cdot)[-1](1),$$

where as usual $[1]$ is the shift of the complexes and (1) is the shift of the gradings.

For $M \in \mathrm{gr} E$, let $\Omega(M)$ be the first syzygy of the minimal graded projective resolution of M . Since E is a graded Frobenius algebra, Ω is an auto-equivalence of the stable category $\underline{\mathrm{gr}} E$. Moreover, $\underline{\mathrm{gr}} E$ is a triangulated category with the shift functor Ω^{-1} . We next show that the triangulated category $\underline{\mathrm{gr}} E$ admits some thick triangulated subcategories. Some terminologies are needed next. Let M be a finite dimensional right graded E -module. Take a minimal graded projective resolution of M as follows

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow M \longrightarrow 0.$$

Recall that the *complexity* of M , denoted by $c(M)$, is the least integer $\lambda \geq 0$ such that $\dim P^{-n} < a n^{\lambda-1}$ for almost all n , where $a > 0$ is a fixed number. The definition of the complexity can be modified even for noninteger λ . Note that $c(M) = 0$ if and only if M is a finite dimensional graded projective E -module (when E is Frobenius). The complexity of a module over a finite dimension algebra is an important invariant in studying the representations of the algebra [Ca, CDW, GLW].

Given a number $\lambda \geq 0$, let \mathcal{C}_λ be the full subcategory of $\underline{\mathrm{gr}} E$ consisting of objects M with $c(M) \leq \lambda$. Similar to [CDW, Proposition 1.3], we have the following lemma, whose proof is almost the same as that of [CDW, Proposition 1.3].

Lemma 4.4. *The induced category \mathcal{C}_λ is a thick triangulated subcategory of $\underline{\mathrm{gr}} E$.*

Proof. Let $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in $\mathrm{gr} E$. Then one sees $c(M) \leq \max\{c(K), c(N)\}$. Hence $M \in \mathcal{C}_\lambda$ if K and N are in \mathcal{C}_λ . Moreover, it follows from [AR, Lemma 2.5] or [GLW, Lemma 2.7] that if K and M (resp. M and N) are in \mathcal{C}_λ , then so is N (resp. K). If $U \rightarrow V \rightarrow Q \rightarrow \Omega^{-1}(U)$ is a triangle in $\underline{\mathrm{gr}} E$ with $U, V \in \mathcal{C}_\lambda$, then both Q and $\Omega^{-1}U$ are in \mathcal{C}_λ , where $\Omega^{-1}(U)$ is the first syzygy of U by taking minimal graded injective resolution of U . Hence \mathcal{C}_λ is a triangulated subcategory of $\underline{\mathrm{gr}} E$. Now if N is a direct summand of M in $\underline{\mathrm{gr}} E$, then N is a direct summand of $\bar{M} \oplus P$ in $\mathrm{gr} E$ for some finite dimensional projective module P . Thus, we have $c(N) \leq c(M)$. Therefore \mathcal{C}_λ is thick. \square

Let $D^b(\mathrm{proj} E)$ be the full subcategory of $D^b(\mathrm{gr} E)$ consisting of all bounded complexes of f.g. graded projective E -modules. Then $D^b(\mathrm{proj} E)$ is equivalent to the triangulated subcategory of $D^b(\mathrm{gr} E)$ generated by E . Since E is a graded Frobenius algebra, there is a well known equivalence of triangulated categories [Be, Corollary 3.9(c)]

$$(E4.4.1) \quad \Phi : D^b(\mathrm{gr} E) / D^b(\mathrm{proj} E) \longrightarrow \underline{\mathrm{gr}} E,$$

where the functor Φ is defined in the following way. Let X^\cdot be an object in $D^b(\mathrm{gr} E)$. We write \bar{X}^\cdot for its image in $D^b(\mathrm{gr} E) / D^b(\mathrm{proj} E)$. Take a bounded above f.g. projective resolution P^\cdot of X^\cdot . Since X^\cdot is bounded, we see that there is an integer n such that the cohomology $H^i P^\cdot = 0$ for all $i \leq -n$. Now, let M be the n th syzygy of the complex P^\cdot . Define $\Phi(\bar{X}^\cdot) = \Omega^{n-1}(M)$ [Or, Sect. 1.3].

One can easily extend the notion of the complexity to bounded complexes of modules over a finite dimensional algebra. Let $\Lambda := \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ be a finite dimensional graded algebra such that Λ_0 is a semisimple algebra. Let $K^-(\text{gr } \Lambda)$ be the homotopy category of bounded above complexes of f.g. right graded Λ -modules. A bounded above complex of graded projective modules $P^\cdot \in K^-(\text{gr } \Lambda)$ is called *minimal* if imd^n is superfluous, namely, $\text{imd}^n \subseteq P^n \Lambda_{\geq 1}$ for all n , where $\Lambda_{\geq 1} = \Lambda_1 \oplus \Lambda_2 \oplus \cdots$. Note that, for any bounded complex X^\cdot of f.g. right graded Λ -modules, there is a minimal complex of graded projective modules P^\cdot that is quasi-isomorphic to X^\cdot . Such P^\cdot is called a *minimal projective resolution* of X^\cdot . It is known that minimal projective resolution of a bounded complex of f.g. graded Λ -module is unique up to isomorphism. We define the *complexity* $c(X^\cdot)$ of X^\cdot to be the least integer λ such that $\dim P^{-n} \leq a n^{\lambda-1}$ for almost all $n \geq 0$. If two bounded complexes X^\cdot and Y^\cdot are quasi-isomorphic, then their minimal projective resolutions are isomorphic. Hence $c(X^\cdot) = c(Y^\cdot)$. Hence, it makes sense to define the complexity of an object in $D^b(\text{gr } A)$. For $\lambda \geq 0$, we denote by $D_\lambda^b(\text{gr } \Lambda)$ the full subcategory of $D^b(\text{gr } \Lambda)$ consisting of objects X^\cdot such that $c(X^\cdot) \leq \lambda$. One sees that $D_0^b(\text{gr } \Lambda)$ is equivalent to $D^b(\text{proj } \Lambda)$.

Lemma 4.5. *Retain the above notation. Then $D_\lambda^b(\text{gr } \Lambda)$ is a thick triangulated subcategory of $D^b(\text{gr } \Lambda)$ for all $\lambda \geq 0$.*

Proof. Take $X^\cdot, Y^\cdot \in D_\lambda^b(\text{gr } \Lambda)$. Let $f : X^\cdot \rightarrow Y^\cdot$ be a morphism in $D^b(\text{gr } \Lambda)$, and $X^\cdot \rightarrow Y^\cdot \rightarrow Z^\cdot \rightarrow X^\cdot[1]$ be a triangle. Let P^\cdot and Q^\cdot be minimal projective resolutions of X^\cdot and Y^\cdot respectively. The morphism f induces a morphism $g : P^\cdot \rightarrow Q^\cdot$. Then $\text{cone}(g) \cong Z^\cdot$ in $D^b(\text{gr } \Lambda)$. Hence the minimal projective resolution of Z^\cdot is a direct summand of $\text{cone}(g)$. Therefore, $c(Z^\cdot) \leq c(\text{cone}(g)) \leq \lambda$, and hence $D_\lambda^b(\text{gr } \Lambda)$ is a triangulated subcategory of $D^b(\text{gr } \Lambda)$. It is clear that $D_\lambda^b(\text{gr } \Lambda)$ is closed under taking direct summands. \square

We can also extend the notion of the GK-dimension from modules to bounded complexes. Let S be a noetherian locally finite graded algebra. Let X^\cdot be a bounded complex of right graded S -modules. Set $H^\cdot(X^\cdot) = \bigoplus_{n \in \mathbb{Z}} H^n(X^\cdot)$, the total cohomology of X^\cdot . Define the GK-dimension of X^\cdot as

$$\text{GKdim } X^\cdot := \text{GKdim } H^\cdot(X^\cdot).$$

By definition, if two complexes X^\cdot and Y^\cdot are quasi-isomorphic, then $\text{GKdim } X^\cdot = \text{GKdim } Y^\cdot$. Hence it makes sense to say the GK-dimension of an object in the bounded derived category. For $\lambda \geq 0$, let $D_{\text{gr } \lambda}^b(\text{gr } S)$ be the full subcategory of $D^b(\text{gr } S)$ consisting of objects X^\cdot such that $\text{GKdim } X^\cdot \leq \lambda$. One sees that $X^\cdot \in D_{\text{gr } \lambda}^b(\text{gr } S)$ if and only if $\text{GKdim } H^n(X^\cdot) \leq \lambda$ for all $n \in \mathbb{Z}$. Hence we have the following.

Lemma 4.6. *For every $\lambda \geq 0$, $D_{\text{gr } \lambda}^b(\text{gr } S)$ is a thick triangulated subcategory of $D^b(\text{gr } S)$.*

We now go back to the H -action on a noetherian connected graded AS regular algebra R .

Setup 4.7. Throughout the rest of this section, we assume that R is a noetherian connected graded AS regular algebra which is also a Koszul algebra. Let B be the graded basic algebra of $R \# H$. Recall that \mathbb{k} is assumed to be algebraically closed.

Then $B_0 \cong \mathbb{k}^{\oplus w}$ for some $w > 0$ [Lemma 4.1(4)]. Let $E = \bigoplus_{n \geq 0} \underline{\text{Ext}}_B^n(B_0, B_0)$ as given in (E4.0.1).

Lemma 4.8. *Let M be a right Koszul B -module. Then $\text{GKdim}(M) = c(K(M))$.*

Proof. Since B is a Koszul algebra, $K(M)$ is a Koszul E -module, moreover, $K(M)$ has a minimal projective resolution [GM, Proposition 5.1]

$$\cdots \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

such that $P^{-n} \cong M_n^* \otimes_{E_0} E$, where $M_n^* = \text{Hom}(M_n, \mathbb{k})$.

Assume $c(K(M)) = \lambda$. Then $\dim M_n \leq \dim P^{-n} \leq an^{\lambda-1}$ for some fixed number $a > 0$ and for all $n \gg 0$. Hence $\text{GKdim}(M) \leq \lambda$.

Conversely, assume $\text{GKdim}(M) = \lambda$. By assumption B is the basic algebra of $R\#H$. Hence there is a finitely generated graded progenerator $R\#H$ -module Q , where $Q := \bigoplus_{i=1}^w Q_i$ as given in (E4.0.2), such that $\underline{\text{End}}_{R\#H}(Q) \cong B$. Write $F = -\otimes_{R\#H} Q^{-1}$. Assume N is a graded right $R\#H$ -module such that $F(N) \cong M$. Then $\text{GKdim}(N) = \text{GKdim}(M) = \lambda$ by Remark 1.5(3). Note that N is also a graded right R -module and by assumption R is a noetherian connected graded AS regular algebra. By [ATV, Proposition 2.21] and [StZ, Corollary 2.2], we have $\dim N_n \leq an^{\lambda-1}$ for some fixed number $a > 0$ and for all $n \gg 0$. Since $M = F(N)$, there is an $a_1 > 0$ such that $\dim M_n \leq a_1 \dim N_n \leq a_1 an^{\lambda-1}$ for all $n \gg 0$. Assume $\dim E = b$. Then $\dim P^{-n} \leq b \dim M_n \leq ba_1 an^{\lambda-1}$. Hence $c(K(M)) \leq \lambda$. \square

Lemma 4.9. *For every integer $\lambda \geq 0$, the Koszul duality given in Proposition 4.3 restricts to the following anti-equivalence of triangulated categories,*

$$D_{\text{gr}\lambda}^b(\text{gr } B) \cong D_{\lambda}^b(\text{gr } E).$$

Proof. Firstly, notice that if M is a f.g. Koszul graded B -module, then, by Proposition 4.3, $H^i(K(M)) = 0$ for $i \neq 0$. In particular, $K(B_0) \cong E$. Hence we see that the thick triangulated subcategory of $D^b(\text{gr } B)$ generated by B_0 is anti-equivalent to the thick triangulated subcategory of $D^b(\text{gr } E)$ generated by E . The former is equivalent to $D_{\text{gr}_0}^b(\text{gr } B)$ and the latter is equivalent to $D^b(\text{proj } E)$ which is in turn equivalent to $D_0^b(\text{gr } E)$. So we have proven the assertion for $\lambda = 0$.

Next we assume that $\lambda \geq 1$. By the Koszul duality, we only need to show the following:

- (a) If $X^\cdot \in D_{\text{gr}\lambda}^b(\text{gr } B)$, then $K(X^\cdot) \in D_{\lambda}^b(\text{gr } E)$.
- (b) If $Y^\cdot \in D_{\lambda}^b(\text{gr } E)$, then $K^{-1}(Y^\cdot) \in D_{\text{gr}\lambda}^b(\text{gr } B)$.

Since $D_{\text{gr}\lambda}^b(\text{gr } B)$ and $D_{\lambda}^b(\text{gr } E)$ are thick triangulated subcategories, it suffices to show that,

- (c) for every f.g. right graded B -modules M with $\text{GKdim} \leq \lambda$, we have $K(M) \in D_{\lambda}^b(\text{gr } E)$, and
- (d) for every f.g. right graded E -modules N such that $c(N) \leq \lambda$, we have $K^{-1}(N) \in D_{\text{gr}\lambda}^b(\text{gr } B)$.

Since B is a Koszul AS-regular algebra, it follows that for sufficiently large k the graded B -module $M_{\geq k}(k)$, where $M_{\geq k} = M_k \oplus M_{k+1} \oplus \cdots$, has linear resolution (see [Jo1, Theorem 3.1], note that it is assumed that the algebra is connected graded in [Jo1], however, the same proof holds in our case), that is, $M_{\geq k}(k)$ is a Koszul module. Consider the exact sequence

$$0 \rightarrow M_{\geq k} \rightarrow M \rightarrow M/M_{\geq k} \rightarrow 0.$$

We see that $\mathrm{GKdim} M_{\geq k} = \mathrm{GKdim} M \leq \lambda$ and

$$K(M/M_{\geq k}) \rightarrow K(M) \rightarrow K(M_{\geq k}) \rightarrow K(M/M_{\geq k})[1]$$

is a triangle in $D^b(\mathrm{gr} E)$. Since $M/M_{\geq k}$ is finite dimensional, $K(M/M_{\geq k}) \in D_0^b(\mathrm{gr} E)$. Since $M_{\geq k}(k)$ is a Koszul module, $c(K(M_{\geq k})) = \mathrm{GKdim}(M_{\geq k}) \leq \lambda$ by Lemma 4.8. Hence $K(M_{\geq k}) \in D_\lambda^b(\mathrm{gr} E)$. Therefore $K(M) \in D_\lambda^b(\mathrm{gr} E)$. This verifies claim (c).

For $Y^\cdot \in D_\lambda^b(\mathrm{gr} E)$, let P^\cdot be a minimal projective resolution of Y^\cdot . Then $\dim P^{-n} \leq a n^{\lambda-1}$ for $n \gg 0$. Since B_0 is a direct sum of finitely many copies of \mathbb{k} , so is E_0 . Assume that $E_0 = \mathbb{k} \oplus \cdots \oplus \mathbb{k}$ is the sum of s copies of \mathbb{k} , and that e_1, \dots, e_s are the corresponding primitive idempotents. Then $E = \bigoplus_{i=1}^s E e_i$. Assume $r = \min\{\dim E e_i \mid i = 1, \dots, s\}$. By the minimality of P^\cdot , we see that

$$\dim P^{-n} \geq r \sum_{j \in \mathbb{Z}} \dim \mathrm{Hom}_{D^b(\mathrm{gr} E)}(Y^\cdot, E_0[n](j)).$$

Hence

$$\sum_{j \in \mathbb{Z}} \dim \mathrm{Hom}_{D^b(\mathrm{gr} E)}(Y^\cdot, E_0[n](j)) \leq \frac{a}{r} n^{\lambda-1}.$$

Given i and j , in the following computations, note that $K(B) \cong E_0$ in $D^b(\mathrm{gr} E)$,

$$\begin{aligned} \dim H^i K^{-1}(Y^\cdot)_j &= \dim \mathrm{Hom}_{D^b(\mathrm{gr} B)}(B[-i](-j), K^{-1}(Y^\cdot)) \\ &= \dim \mathrm{Hom}_{D^b(\mathrm{gr} E)}(Y^\cdot, K(B[-i](-j))) \\ &= \dim \mathrm{Hom}_{D^b(\mathrm{gr} E)}(Y^\cdot, E_0[i+j](-j)) \\ &\leq \frac{a}{r} (i+j)^{\lambda-1}. \end{aligned}$$

Hence $\mathrm{GKdim} H^i K^{-1}(Y^\cdot) \leq \lambda$ for all i . Therefore, $K^{-1}(Y^\cdot) \in D_{\mathrm{gr} \lambda}^b(\mathrm{gr} B)$. Thus we verify claim (b) (and (d)). \square

Lemma 4.10. *There are equivalences of triangulated categories:*

$$D_\lambda^b(\mathrm{gr} E)/D^b(\mathrm{proj} E) \cong \mathcal{C}_\lambda,$$

for all $\lambda \geq 0$.

Proof. By Lemma 4.5, $D_\lambda^b(\mathrm{gr} E)$ is a thick triangulated subcategory of $D^b(\mathrm{gr} E)$ for each $\lambda \geq 0$. One sees that $D_\lambda^b(\mathrm{gr} E)/D^b(\mathrm{proj} E)$ is a full triangulated subcategory of $D^b(\mathrm{gr} E)/D^b(\mathrm{proj} E)$ for each $\lambda \geq 1$. Consider the equivalence

$$\Phi : D^b(\mathrm{gr} E)/D^b(\mathrm{proj} E) \longrightarrow \underline{\mathrm{gr} E}$$

as established by (E4.4.1). By the definition of Φ , one easily sees that if $c(X^\cdot) \leq \lambda$ then $c(\Phi(X^\cdot)) \leq \lambda$. Hence Φ sends objects in $D_\lambda^b(\mathrm{gr} E)/D^b(\mathrm{proj} E)$ to objects in \mathcal{C}_λ . If N is a right graded E -module such that $c(N) \leq \lambda$, then we may view N as an object in $D_\lambda^b(\mathrm{gr} E)/D^b(\mathrm{proj} E)$, and we have $\Phi(N) \cong N \in \mathcal{C}_\lambda$. Therefore the restriction of Φ to $D_\lambda^b(\mathrm{gr} E)/D^b(\mathrm{proj} E)$ is essential dense onto \mathcal{C}_λ , and hence $D_\lambda^b(\mathrm{gr} E)/D^b(\mathrm{proj} E)$ is equivalent to \mathcal{C}_λ . \square

We are ready to prove the main result of this section, which may be viewed as a generalization of the classical BGG correspondence [BGG, Jo2, Mor1].

Theorem 4.11. *Let B and E be as in Setup 4.7. Then, for each $\lambda \geq 0$, there is an anti-equivalences of triangulated categories*

$$D^b(\text{qgr}_\lambda B) \cong \underline{\text{gr}}E/\mathcal{C}_\lambda.$$

Proof. By the Koszul duality in Proposition 4.3, we have an anti-equivalence $K : D^b(\text{gr } B) \rightarrow D^b(\text{gr } E)$. Lemma 4.9 and its proof show that K restricts to anti-equivalences $D_{\text{gr}_\lambda}^b(\text{gr } B) \cong D_\lambda^b(\text{gr } E)$ for each $\lambda \geq 0$. Hence we obtain anti-equivalences

$$D^b(\text{gr } B)/D_{\text{gr}_\lambda}^b(\text{gr } B) \cong D^b(\text{gr } E)/D_\lambda^b(\text{gr } E)$$

for each $\lambda \geq 0$. By the equivalence $\Phi : D^b(\text{gr } E)/D^b(\text{proj } E) \rightarrow \underline{\text{gr}}E$ and Lemma 4.10, we have

$$D^b(\text{gr } E)/D_\lambda^b(\text{gr } E) \cong \underline{\text{gr}}E/\mathcal{C}_\lambda.$$

On the other hand, by [Mi, Theorem 3.2] we have the equivalence of triangulated categories

$$D^b(\text{gr } B)/D_{\text{gr}_\lambda}^b(\text{gr } B) \cong D^b(\text{qgr}_\lambda B),$$

since $\text{gr}_\lambda B$ is a Serre subcategory of $\text{gr } B$. Combine the (anti-)equivalences above, we finally obtain the anti-equivalences as required. \square

Remark 4.12. In the case when $\lambda = 0$, we have $\mathcal{C}_0 = 0$ and $\text{qgr}_0 B = \text{qgr } B$, the quotient category of $\text{gr } B$ modulo by the Serre subcategory of finite dimensional graded B -modules. Then the anti-equivalence in the theorem above has the form

$$D^b(\text{qgr } B) \cong \underline{\text{gr}}E$$

which is the classical BGG correspondence established in [BGG, Jo2, Mor1].

Corollary 4.13. *Assume Setup 4.7. Let R be $CM(2)$ of $\text{GKdim } d \geq 2$. If $R\#H \cong \text{End}_{R^H}(R)$, then there is an canonical anti-equivalence of triangulated categories*

$$D^b(\text{qgr}_{d-2} R^H) \cong \underline{\text{gr}}E/\mathcal{C}_{d-2}$$

where $E = \bigoplus_{n \geq 0} \text{Ext}_{R\#H}^n(H, H)$.

Proof. Note that R and H are in the situation as in Theorem 3.5.

Assume that $R\#H \cong \text{End}_{R^H}(R)$, we will show the category equivalence. Take B to be the basic algebra of $R\#H$. Then $R\#H$ and B are graded Morita equivalence. Since R is a Koszul algebra, E and $E' := \bigoplus_{n \geq 0} \text{Ext}_B^n(B_0, B_0)$ are also graded Morita equivalence [Lemma 4.1(2)]. Since the graded Morita equivalence preserves the GK-dimension, we have that $\text{gr}_\lambda(R\#H)$ and $\text{gr}_\lambda B$ are equivalent. As a consequence, $D^b(\text{qgr}_\lambda B)$ is equivalent to $D^b(\text{qgr}_\lambda R\#H)$. Combining this with Theorems 3.5 and 4.11, we have

$$D^b(\text{qgr}_{d-2} R^H) \cong D^b(\text{qgr}_{d-2} R\#H) \cong D^b(\text{qgr}_{d-2} B) \cong \underline{\text{gr}}E'/\mathcal{C}_{d-2} \cong \underline{\text{gr}}E/\mathcal{C}_{d-2}.$$

\square

Theorem 0.7 follows from Lemma 4.1, Theorem 4.11 and Corollary 4.13.

5. PROOF OF THEOREM 0.5

In this section let \mathbb{k} be a field and $\text{char } \mathbb{k} \nmid 2n$. We consider some finite group actions on the (-1) -skew polynomial algebra $\mathbb{k}_{-1}[x_1, \dots, x_n]$ and prove Theorems 0.5. We need a few lemmas.

Let R be a graded domain and let H be a dual group $(\mathbb{k}G)^\circ$ acting on R homogeneously. Assume that the H -action on R is *inner faithful*. If R is generated in degree 1, then this is equivalent to that H -action on R_1 is inner faithful. In this setting, R is also $\mathbb{Z} \times G$ -graded. Forgetting the \mathbb{Z} -grading, R is G -graded and R is not G' -graded for any proper subgroups $G' \subsetneq G$. Write $R = \bigoplus_{g \in G} R_g$. Part (5) of the next lemma is due to [CKWZ2]. Let I be the ideal (e) in $B := R \# H$ where $e = 1 \# \int$ and \int is the integral of H such that $\epsilon(\int) = 1$.

Lemma 5.1. *Let R be a connected graded domain. Let $H := (\mathbb{k}G)^\circ$ act on R inner faithfully and write $R = \bigoplus_{g \in G} R_g$.*

- (1) *For every $g \in G$, $R_g \neq 0$.*
- (2) *R is a free H -module.*
- (3) *If $f \in \bigcap_{g \in G} RR_g$, then $f \# 1 \in I$.*
- (4) [CKWZ2] *Suppose $f_1, \dots, f_m \in R$ are G -homogeneous elements in R such that the set $\{h_w\}_{w=1}^m = G$ where $h_w := \deg_G(f_w) \cdots \deg_G(f_m)$. Then $f_1 \cdots f_m \# 1 \in I$.*
- (5) *Suppose $d := \text{GKdim } R < \infty$. Then $R \# H$ is prime and the algebra map*

$$R \# H \rightarrow \text{End}_{R^H}(R)$$

is injective. As a consequence, $\mathfrak{p}(R, H) \geq 1$.

- (6) *$\mathfrak{p}(R, H) \geq d - \text{GKdim } R / (\bigcap_{g \in G} RR_g) \geq d - \max\{\text{GKdim } R / RR_g \mid g \in G\}$.*
- (7) *If there is an integer s such that $RR_g \supseteq R_{\geq s}$ for all $g \in G$, then $\mathfrak{p}(R, H) = d$.*

Proof. (1) Let $X = \{g \in G \mid R_g \neq 0\}$. Since R is a domain, X is a sub-semigroup of G . Since X is finite, it is a group. By the inner faithfulness, $X = G$.

(2) We may assume $R \neq \mathbb{k}$. Since R is a connected graded domain, R is infinite dimensional. As a consequence, R^H is infinite dimensional. So each R_g is infinite dimensional. So R is a free H -module of infinite rank.

(3) Let $\{g_1, \dots, g_m\}$ be the set G where g_1 is the unit of G . The dual basis $\{p_{g_1}, \dots, p_{g_m}\}$ is the complete set of orthogonal idempotents of H that forms a \mathbb{k} -linear basis of H . The integral of H is p_{g_1} and $\sum_{i=1}^m p_{g_i}$ is the algebra unit of H . The coproduct of H is determined by

$$\Delta(p_g) = \sum_{h \in H} p_h \otimes p_{h^{-1}g}$$

for all $g \in G$. Let f be a G -homogeneous element of degree $g \in G$. Then

$$e(f \# 1) = (1 \# p_{g_1})(f \# 1) = \sum_h p_h(f) \# p_{h^{-1}} = p_g(f) \# p_{g^{-1}} = f \# p_{g^{-1}}.$$

This implies that

$$f \# p_{g^{-1}} \in I$$

for all $f \in R_g$. Since I is a left R -module, we have

$$RR_g \# p_{g^{-1}} \subseteq I.$$

If $f \in \bigcap_{g \in G} RR_g$, then $f \# p_{g^{-1}} \in I$ for all $g \in G$. Then $f \# 1 = \sum_{g \in G} f \# p_{g^{-1}} \in I$.

(4) Let $h_w = \deg_G(f_w) \cdots \deg_G(f_m)$ and let $f = f_1 \cdots f_m$. Then $f_w \cdots f_m \in R_{h_w}$ and $f \in RR_{h_w}$ for all w . By hypothesis $\{h_w\}_{w=1}^m$ equals G . Then $f \in \bigcap_{g \in G} RR_g$. The assertion follows by part (3).

(5) By part (3,4), there is a nonzero element $f \in R$ such that $f \# 1 \in I$. Then $B/(I)$ has GK-dimension no more than the $\text{GKdim } R - 1$, or equivalently, $\text{p}(R, H) \geq 1$. The assertion follows by Lemma 3.10(2,4).

(6) This follows from Lemma 5.2 below and the fact that the image of $R \rightarrow R \# H/(I)$, denoted by R' , is a quotient of $R/(\bigcap_{g \in G} RR_g)$, by part (3), and that there is an embedding $R/(\bigcap_{g \in G} RR_g) \rightarrow \bigoplus_{g \in G} R/(RR_g)$.

(7) This is an immediate consequence of (6). \square

Let T be a subalgebra of R such that R_T and ${}_T R$ are f.g..

Lemma 5.2. *Let R' be the image of the map $R \rightarrow B \rightarrow B/I$ and T' be the image of T in R' . Then $\text{GKdim } T' = \text{GKdim } R' = \text{GKdim } B/I$.*

Proof. This follows from the fact that B/I is f.g. over R' . \square

Aiming for a proof of Theorem 0.5, we now let R be the (-1) -skew polynomial ring $\mathbb{k}_{-1}[x_1, \dots, x_n]$. Let H be the group algebra $\mathbb{k}W$ where W is generated by the permutation $\sigma : x_i \rightarrow x_{i+1}, x_n \rightarrow x_1$ for all $1 \leq i \leq n-1$. Theorem 0.5 asserts that $R \# \mathbb{k}W \cong \text{End}_{R^W}(R)$.

Let ξ be a fixed primitive n th root of unity. We might assume that $\xi \in \mathbb{k}$ since $\text{p}(R, H)$ is preserved by field extensions. Let $y_j = \sum_{i=1}^n \xi^{ij} x_i$. Then $\{y_1, \dots, y_n\}$ is a basis of R_1 . It is clear that $\sigma(y_j) = \xi^{-j} y_j$. Let T be the central subalgebra of R generated by $\{x_i^2\}_{i=1}^n$ and let $Y_j = \sum_{i=1}^n \xi^{ij} x_i^2$. Then $\{Y_i\}_{i=1}^n$ is a basis of the vector space $\bigoplus_{i=1}^n \mathbb{k}x_i^2$. It is clear that R is a f.g. module over the central subalgebra T .

The group algebra $\mathbb{k}W$ can be viewed as a $(\mathbb{k}G)^\circ$ where G is the character group of W . Note that $G \cong W$ as abstract groups. Let g be a generator of G such that $\deg_G(y_1) = g$. Then $\deg_G(y_i) = g^i$ for all $0 \leq i \leq n-1$.

Let $U_n = \{(0, 1), (1, 2), \dots, (\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor)\}$. For any integer i , let \bar{i} be the integer between 0 and $n-1$ such that $\bar{i} \equiv i \pmod{n}$. The following lemma is easy to check.

Lemma 5.3. *Retain the above notation for R . Let i, j, k be integers (between 0 and $n-1$ modulo n).*

- (1) $y_i y_j + y_j y_i = 2Y_{i+j} = 2Y_{\bar{i}+\bar{j}}$.
- (2) $y_k y_j y_i = -y_k y_i y_j + (y_i y_j + y_j y_i) y_k$.
- (3) $R'/(Y_i \mid i = 0, \dots, n-1)$ is finite dimensional.
- (4) If $i \geq \frac{n}{2}$, $y_i^2 = y_j^2$ where $j = i - \frac{n}{2}$.
- (5) Assume n is even. If $i < j$ and $(i, j) \notin U_n$

$$y_j y_i = \begin{cases} -y_i y_j + 2y_s^2 & i+j = \text{even and } < n, s = \frac{i+j}{2}, \\ -y_i y_j + 2y_s^2 & i+j = \text{even and } \geq n, s = \frac{i+j-n}{2}, \\ -y_i y_j + (y_s y_{s+1} + y_{s+1} y_s) & i+j = \text{odd and } < n, s = \frac{i+j-1}{2}, \\ -y_i y_j + (y_s y_{s+1} + y_{s+1} y_s) & i+j = \text{odd and } > n, s = \frac{i+j-1-n}{2}. \end{cases}$$

- (6) Suppose $n = 2^d$. For each s , $(y_s y_{s-1})^n = 0$ in R' , and hence in $R'' := R'/(y_i^2 \mid i = 0, \dots, \frac{n}{2})$.
- (7) Suppose $n = 2^d$. Then R'' is finite dimensional.
- (8) Suppose $n = 2^d$. Then $y_s^{2^{d+1}} = 0$ in R' for all s .

Proof. (1) By a direct computation.

(2) This follows from part (1) and the fact $Y_{\frac{1}{i+j}}$ is central in R .

(3) This follows from Lemma 5.2 and the fact $T/(Y_i \mid i = 0, \dots, n-1) = k$.

(4) Follows from part (1).

From now on we assume that n is even.

(5) It follows from part (1).

(6) Since $\deg_G(y_s y_{s-1})$ is odd, which is coprime to n . The assertion follows from Lemma 5.1(4).

(7) We use Diamond lemma. Fix an order $y_0 < y_1 < \dots < y_{n-2} < y_{n-1}$ in R'' , and extend it to the lexicographical order on all monomials.

By part (5), for any $j > i$, $y_j y_i$ can be written as a linear combination of lower terms. By part (4), y_i^2 when $i \geq \frac{n}{2}$ can be expressed as a lower term. By part (2), if $k > j > i$, $y_k y_j y_i$ can be written as a linear combination of lower terms.

Let X be a reduced monomial basis of R'' . Let $0 \neq f = y_{i_1} y_{i_2} \dots y_{i_h} \in X$. Then $i_\alpha \neq i_{\alpha+1}$ since $y_i^2 = 0$ in R'' for all i . Since there are only n possible i_α , the maximal length of increasing i_α is at most n . If $i_\alpha > i_{\alpha+1}$, then, by part (5), $i_\alpha \leq \frac{n}{2}$ and $i_{\alpha+1} = i_\alpha - 1$. Suppose we have two pairs $(i_\alpha, i_{\alpha+1})$ and $(i_\beta, i_{\beta+1})$ in U_n with $\alpha < \beta$. By part (2), $\beta \neq \alpha + 1$. If $\beta > \alpha + 1$, then i_s are strictly increasing between $i_{\alpha+1}$ and i_β . So $i_\beta \geq i_\alpha + 1$. If $\beta = \alpha + 1$, then $i_\beta \geq i_\alpha$. By part (6), $(y_{i_\alpha} y_{i_{\alpha-1}})^n = 0$ in R'' . This implies that there are at most $\frac{1}{2}n^2$ pairs $(i_\alpha, i_{\alpha+1}) \in U_n$ appeared in f . Hence the degree of f is uniformly bounded. This means that X is finite and R'' has finite dimension.

(8) By part (4), we may assume $s < \frac{n}{2}$. If s is odd (which is coprime to n), then, by Lemma 5.1(4), $y_s^n = 0$ in R' .

Now we consider general s . By Lemma 5.1(3), it suffices to show that $y_s^{2(d+1)} \in RR_{g^i}$ for any $i = 1, \dots, n-1$. Note that $y_s^{2(d+1)} \in R = RR_{g^0}$ is clear. Fix i and write \equiv to be $\equiv \pmod{RR_{g^i}}$. We start with $y_i \equiv 0$. Then

$$(E5.3.1) \quad 2y_s^2 = y_i y_{2s-i} + y_{2s-i} y_i \equiv y_i y_{2s-i}.$$

We prove by induction that, for each w ,

$$(E5.3.2) \quad (2y_s)^{2w} \equiv y_{j_1} y_{j_2} \dots y_{j_w} y_{i_w} \dots y_{i_2} y_{i_1}$$

where

$$i_\alpha = 2^\alpha s - 2^{\alpha-1} i, \quad \text{and} \quad j_\alpha = -2^\alpha s + 2^{\alpha-1} i + 2s.$$

When $w = 1$, this is (E5.3.1). Suppose (E5.3.2) holds for $w-1 \geq 1$. Then

$$\begin{aligned} (2y_s)^{2w} &= (2y_s^2)(2y_s)^{2(w-1)} \equiv (2y_s^2)y_{j_1} y_{j_2} \dots y_{j_{w-1}} y_{i_{w-1}} \dots y_{i_2} y_{i_1} \\ &\equiv y_{j_1} y_{j_2} \dots y_{j_{w-1}} (2y_s^2) y_{i_{w-1}} \dots y_{i_2} y_{i_1} \\ &\equiv y_s^2 y_{j_1} y_{j_2} \dots y_{j_{w-1}} (y_{i_w} y_{j_w} + y_{j_w} y_{i_w}) y_{i_{w-1}} \dots y_{i_2} y_{i_1} \\ &\equiv y_s^2 y_{j_1} y_{j_2} \dots y_{j_{w-1}} (y_{j_w} y_{i_w}) y_{i_{w-1}} \dots y_{i_2} y_{i_1} \end{aligned}$$

since $y_{j_w} y_{i_{w-1}} \dots y_{i_2} y_{i_1} \in R_{g^i}$. Therefore (E5.3.1) follows from the induction. Now take $w = d$, we have

$$\deg_G(y_{i_d} \dots y_{i_1}) = g^{(2^{d+1}-2)s - (2^d-1)i} = g^{-2s+i}$$

which implies that $\deg_G(y_s^2 y_{i_d} \cdots y_{i_1}) = g^{(2^{d+1}-2)s-(2^d-1)i} = g^i$ and

$$\begin{aligned} (2y_s)^{2(d+1)} &\equiv (2y_s^2) y_{j_1} y_{j_2} \cdots y_{j_{d-1}} y_{j_d} y_{i_d} y_{i_{d-1}} \cdots y_{i_2} y_{i_1} \\ &\equiv 2y_{j_1} y_{j_2} \cdots y_{j_{d-1}} y_{j_d} (y_s^2 y_{i_d} y_{i_{d-1}} \cdots y_{i_2} y_{i_1}) \equiv 0. \end{aligned}$$

Since $\text{char } \mathbb{k} \nmid 2n$, $y_s^{2(d+1)} \in RR_{g^i}$ for all i as desired. \square

Proposition 5.4. *Let R be the (-1) -skew polynomial ring $\mathbb{k}_{-1}[x_1, \dots, x_n]$ for $n \geq 2$. Let H be the group algebra $\mathbb{k}W$ where W is generated by the permutation $\sigma : x_i \rightarrow x_{i+1}, x_n \rightarrow x_1$ for all $1 \leq i \leq n-1$. Suppose $n = 2^d$ for some $d \geq 1$.*

- (1) $\mathfrak{p}(R, H) = \text{GKdim } R = n \geq 2$.
- (2) $\text{qgr}_0 R \# H \cong \text{qgr}_0 R^H$.
- (3) R^H has graded isolated singularities in the sense of [Ue, Definition 2.2].

Proof. (1) By Lemma 5.2, it suffices to show that $\text{GKdim } R' = 0$. Let \mathfrak{p} be a minimal prime ideal of R' and let $T = R'/\mathfrak{p}$. For any s , y_s^2 is normal in R' . By Lemma 5.3(8), $(y_s^2)^{(d+1)} = 0$ in R' . Since $T = R'/\mathfrak{p}$ is prime, $y_s^2 = 0$ in T . By Lemma 5.3(7), T is finite dimensional (and hence $T \cong k$). Therefore R' is finite dimensional and $\text{GKdim } R' = 0$ as desired.

(2) This is a consequence of Theorem 0.6.

(3) This follows from the definition and part (2). \square

Remark 5.5. We have some comments concerning the above computation.

- (1) It is a little bit surprising that if $n = 2^d$, then $\mathfrak{p}(R, H) = \text{GKdim } R$. However conjecture 0.9 would give an explanation of this phenomena.
- (2) We conjecture that when $n \neq 2^d$, then $\mathfrak{p}(R, H) < \text{GKdim } R$ and R^H does not have graded isolated singularities. It would be nice if there is a formula for $\dim_{\text{sing}}(R^H)$ in terms of n .

In practice, we assume that H acts on A inner faithfully. When R is an AS regular domain, we conjecture that $\mathfrak{p}(R, H) \geq 1$, see Lemma 3.10. By Theorem 3.5, when $\mathfrak{p}(R, H) \geq 2$, then the Auslander theorem holds. By Lemma 2.3, when $\mathfrak{p}(R, H) = \alpha$ (or more generally $\mathfrak{p}(R, H) \geq \alpha$), then $\text{qgr}_{d-\alpha} R \# H \cong \text{qgr}_{d-\alpha} R^H$ where $d = \text{GKdim } R$. Since $R \# H$ is regular, this equivalence gives some bounds on the dimension of singular locus of R^H . Therefore $\mathfrak{p}(R, H)$ relates several other properties and invariants of the H -action on R .

The definition of reflection number $r(R, G)$ is given in Definition 0.8. Recall that G is called *conventionally small* if $r(R, G) \geq 2$. If the Conjecture 0.9 holds, then Auslander theorem holds for classical small groups by Theorem 3.5.

Remark 5.6. (1) By definition, $r(R, G') \geq r(R, G)$ if G' is a subgroup of G .

(2) Suggested by part (1), we conjecture that $\mathfrak{p}(R, H') \geq \mathfrak{p}(R, H)$ if H' is a Hopf subalgebra of H (assuming H' -action on R is still inner faithful).

(3) It is not clear to us if there is any relationship between $\dim_{\text{sing}}(R^{H'})$ and $\dim_{\text{sing}}(R^H)$. Suggested by part (2), we would guess that $\dim_{\text{sing}}(R^{H'}) \leq \dim_{\text{sing}}(R^H)$, but we don't have any result to support the claim.

We can compute some bounds for $\mathfrak{p}(R, G)$ and $r(R, G)$ in the following situation.

Theorem 5.7. *Let R be the (-1) -skew polynomial ring $\mathbb{k}_{-1}[x_1, \dots, x_n]$ for $n \geq 2$. Let H be the group algebra $\mathbb{k}W$ where W is generated by the permutation $\sigma : x_i \rightarrow x_{i+1}, x_n \rightarrow x_1$ for all $1 \leq i \leq n-1$.*

- (1) If $n = 2^d$, then $\mathfrak{p}(R, W) = n$.
- (2) $\mathfrak{r}(R, W) = \begin{cases} n & n = 2^d \\ n(1 - \frac{1}{p}) & p > 1 \text{ is the minimal odd prime factor of } n. \end{cases}$
- (3) $\mathfrak{p}(R, W) \geq \begin{cases} \phi(n) & n = \text{odd, or } n = 2m \text{ with } m > 1 \text{ an odd number} \\ \frac{\phi(n)}{2} & n = \text{otherwise} \end{cases}$
where $\phi(n) := n \prod_{\text{all primes } p \mid n} (1 - \frac{1}{p})$ is the Euler's totient function.
- (4) If $n = p^d$ where p is a prime number, then

$$\mathfrak{p}(R, W) \geq \mathfrak{r}(R, W) = \begin{cases} n & p = 2, \\ n(1 - \frac{1}{p}) & p > 2 \end{cases}.$$

As a consequence, Conjecture 0.9 holds in this case.

Proof. (1) This is Proposition 5.4.

(2) Let σ be the cycle permutation sending $x_i \rightarrow x_{i+1}$ for all $i < n$ and sending x_n to x_1 . Let g be σ^i for some $i < n$. Then g is a product of a -many of disjoint cycles of length b where $ab = n$. Let $o(g)$ be the order of the pole of $\text{Tr}(g, t)$ at $t = 1$. By [KKZ2, Lemma 1.7], $o(g)$ is equal to the number of odd cycles. (The authors of [KKZ2] assume that $\text{char } \mathbb{k} = 0$, however, [KKZ2, Lemma 1.7] holds when $\text{char } \mathbb{k} \neq 2$.) Hence $o(g) = a = n/b$ when b is odd. If n is of the form 2^d , then $o(g) = 0$ and $\mathfrak{r}(R, G) = n$. If n contains some odd factor, then the maximal possible $o(g)$ attains when b is the smallest odd prime factor of n . Therefore $\mathfrak{r}(R, G) = n - n/p$ where p is the minimal odd prime factor of n . The assertion follows.

(3) As before, let R' be the image of R in B/I and let T be the subalgebra of R' generated by $\{x_i^2\}_{i=1}^n$. Then T is commutative and $\text{GKdim } T = \text{GKdim } R' = \text{GKdim } B/I$. It suffice to show that

$$\text{GKdim } T/\mathfrak{p} \leq n - \begin{cases} \phi(n) & n = \text{odd, or } n = 2m \text{ with } m > 1 \text{ an odd number} \\ \frac{\phi(n)}{2} & n = \text{otherwise} \end{cases}$$

for all minimal prime \mathfrak{p} of T .

Let ξ be a primitive n th root of unity. For any $0 \leq i \leq n-1$, let $y_i = \sum_{j=1}^n \xi^{ij} x_j$ and $Y_i = \sum_{j=1}^n \xi^{ij} x_j^2$. Since $x_i x_j + x_j x_i = 0$ for all $i \neq j$, we have $y_i^2 = Y_{2i}$. Since x_i^2 are central in R for all i , Y_i are all central in R for all i . Note that the commutative ring T/\mathfrak{p} is generated by $\{x_i^2\}_{i=1}^n$, hence by $\{Y_i\}_{i=1}^n$.

Let i_0 be any positive integer less than n that is coprime to n . The cardinality of $\{Y_{2i_0} \mid \gcd(i_0, n) = 1\}$, denoted by N_n , is $\phi(n)$ when n is odd, or $n = 2m$ with $m > 1$ an odd number; and $\frac{1}{2}\phi(n)$ when $n = 2^k m$ with $k \geq 2$. We give some details in the case when $n = 2m$ where $m > 1$ is an odd integer. It suffices to show that $Y_{2i_0} \neq Y_{2i_1}$ if $\gcd(i_0, n) = \gcd(i_1, n) = 1$ and $1 \leq i_0 < i_1 \leq n$. If $Y_{2i_0} = Y_{2i_1}$, then $n \mid 2(i_1 - i_0)$. This implies that $m \mid (i_1 - i_0)$. Since both i_1 and i_0 are odd (as $\gcd(i_0, n) = \gcd(i_1, n) = 1$), $2m \mid (i_1 - i_0)$, which is impossible. So $N_n = \phi(n)$ in this case.

Suppose $\gcd(n, i_0) = 1$. By Lemma 5.1, $y_{i_0}^n = 0$ in R' (by taking $f_i = y_{i_0}$ for all i). Then $Y_{2i_0}^n = y_{i_0}^{2n} = 0$. Since T/\mathfrak{p} is prime, $Y_{2i_0} = 0$ in T/\mathfrak{p} . Therefore T/\mathfrak{p} is generated by no more than $(n - N_n)$ elements. This implies that $\text{GKdim } T/\mathfrak{p} \leq n - N_n$ and the assertion follows.

(4) This is an immediate consequence of parts (1,2). \square

Now we are ready to prove Theorem 0.5.

Proof of Theorem 0.5. Note that $G = W$. By Theorem 5.7(1,3), $\mathbf{p}(R, W) \geq 2$. It is well-known that skew polynomial rings are CM. Then the final assertion follows from Theorem 0.3. \square

Let S_n act on $\{x_1, \dots, x_n\}$ as permutations. This action extends to an algebraic action on both commutative polynomial ring $\mathbb{k}[x_1, \dots, x_n]$ and the skew polynomial ring $\mathbb{k}_{-1}[x_1, \dots, x_n]$. We define the pertinency for S_n and its subgroups in these cases.

Definition 5.8. Suppose $\text{char } \mathbb{k} = 0$. Let G be a subgroup of S_n that acts naturally on $\mathbb{k}[x_1, \dots, x_n]$ and $\mathbb{k}_{-1}[x_1, \dots, x_n]$ respectively. Define the *positive and negative pertinency* of G to be

$$\mathbf{p}_+(G) = \mathbf{p}(\mathbb{k}[x_1, \dots, x_n], G) \quad \text{and} \quad \mathbf{p}_-(G) = \mathbf{p}(\mathbb{k}_{-1}[x_1, \dots, x_n], G)$$

respectively.

Question 5.9. Is there a combinatorial algorithm to compute $\mathbf{p}_+(G)$ and $\mathbf{p}_-(G)$ for all $G \subseteq S_n$?

The second part of the above question concerning $\mathbf{p}_-(G)$ was suggested by Ellen Kirkman. See [BHZ, Section 7] for more comments.

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